ERRATUM: TAME TOPOLOGY OF ARITHMETIC QUOTIENTS AND ALGEBRAICITY OF HODGE LOCI

B. BAKKER, B. KLINGLER, AND J. TSIMERMAN

Abstract. We correct an error in the functoriality of the $\mathbb{R}_{\mathrm{alg}}$-definable structures on arithmetic quotients $\Gamma\backslash G/M$ constructed in [2]. The statements for Hodge manifolds and period maps are unaffected.

1.1. **Summary of errors.** Arithmetic quotients are real analytic manifolds of the form $S_{\Gamma,G,M} := \Gamma\backslash G/M$, for $\mathbf{G}$ a connected semi-simple linear algebraic $\mathbb{Q}$-group, $G := \mathbf{G}(\mathbb{R})^+$ the real Lie group connected component of the identity of $\mathbf{G}(\mathbb{R})$, $M \subset G$ a connected compact subgroup and $\Gamma \subset \mathbf{G}(\mathbb{Q})^+ := \mathbf{G}(\mathbb{Q}) \cap G$ a neat arithmetic group. Let $\pi : G \to S_{\Gamma,G,M}$ be the quotient map. The construction of an $\mathbb{R}_{\mathrm{alg}}$-definable manifold structure on $S_{\Gamma,G,M}$ in [2]—as well as the Borel–Serre compactification due to Borel–Ji—is correct but depends on a choice of maximal compact $K \supset M$. This dependence is often ignored in [2] which leads to some errors, particularly in the functoriality of the definable structures. The resulting errors are as follows:

- Theorem 1.1 (1) is incorrect, in that $S_{\Gamma,G,M}$ admits a natural $\mathbb{R}_{\mathrm{alg}}$ structure *only* once one fixes a choice of maximal compact K. The characterization by images of Siegel sets is correct, as long as one restricts to Siegel sets associated with K. Likewise, the compatibility with the Borel–Serre compactification is correct as long as one uses the same K for both.
- Theorem 1.1(2) is incorrect; a more restrictive notion of morphisms of arithmetic quotients is needed in order for them to be compatible with the definable structures.
- Definition 2.5 should fix a maximal compact subgroup $K \supset M$, and should read: *A Siegel set for G/M associated to K is the image of a Siegel set of G associated to $\mathbf{P}$ and K for some rational parabolic $\mathbf{P}$, as in Definition 2.3.*
- Proposition 2.7 is correct as long as all Siegel sets are associated to the same maximal compact subgroup K.
- §2.3 is correct as written, but everything depends on the choice of K, and not just the triple (Γ, G, M).

1.2. **Definable structures.** We refer to $S_{\Gamma,G,M,K}$ as the $\mathbb{R}_{\mathrm{alg}}$-definable manifold structure on $S_{\Gamma,G,M}$ associated to K (as in [2, §3.1]). Briefly, for a $\mathbb{Q}$-parabolic $\mathbf{P}$ of $\mathbf{G}$ with unipotent radical $\mathbf{N}$, K determines a real Levi $L \subset G$ which decomposes as $L = AQ$ where A is the center and Q is semi-simple. A semialgebraic Siegel set of G associated to $\mathbf{P}$ and K is then a set of the form $\mathfrak{S} = UaA_{>0}W$ where $U \subset \mathbf{N}(\mathbb{R})$, $W \subset QK$ are

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bounded semialgebraic subsets, $a \in A$, and $A_{>0}$ the cone corresponding to the positive root chamber. By a Siegel set of G associated to K we will mean a semialgebraic Siegel set associated to $\mathbf{P}$ and K for some $\mathbb{Q}$-parabolic $\mathbf{P}$ of $\mathbf{G}$. Then $S_{\Gamma,G,M,K}$ is defined so that the induced map $\pi|_{\mathfrak{S}} : \mathfrak{S} \to S_{\Gamma,G,M,K}$ is $\mathbb{R}_{\mathrm{alg}}$-definable for any Siegel set $\mathfrak{S}$ associated to K.

Remark 1.1.

1. Let $\mathbf{P}$ be a $\mathbb{Q}$-parabolic and $\mathfrak{S}$ a Siegel set associated to $\mathbf{P}$ and K. For $g \in G$, $\mathfrak{S}g$ is a Siegel set associated to $\mathbf{P}$ and $g^{-1}Kg$ since $G = K\mathbf{P}(\mathbb{R})$, while for $g \in \mathbf{P}(\mathbb{R})$, $g\mathfrak{S}$ is a Siegel set associated to $\mathbf{P}$ and K.
2. Let X be the symmetric space of maximal compact subgroups of G. For any $\mathbb{Q}$-parabolic $\mathbf{P}$, the real points $\mathbf{P}(\mathbb{R})$ act transitively by conjugation on X. It follows from the previous remark the natural map $S_{\Gamma,G,K,K} \to S_{\Gamma,G,gKg^{-1},gKg^{-1}}$ given by $\Gamma hK \mapsto \Gamma hKg^{-1}$ is $\mathbb{R}_{\mathrm{alg}}$-definable. In particular, as X is equivariantly identified with (a finite quotient of) G/K upon choosing a basepoint, $\Gamma\backslash X$ has a canonical $\mathbb{R}_{\mathrm{alg}}$-definable structure.
3. We may alternatively define a G-space as a topological space Y with a continuous transitive left G-action and connected compact stabilizers. Upon choosing a basepoint we equivariantly identify $Y \simeq G/M$. A choice of equivariant map $Y \to X$ then endows the quotient $\Gamma\backslash Y$ with an $\mathbb{R}_{\mathrm{alg}}$-definable structure: a definable fundamental set for Y is obtained as the pullback of a definable fundamental set of X.

1.3. **Functoriality.** Recall that a choice of maximal compact K is equivalent to a Cartan involution θ_K of the Lie algebra $\mathfrak{g}$. We define a morphism of arithmetic quotients with a choice of maximal compact to be a real semialgebraic map $(\phi, g) : S_{\Gamma',G',M',K'} \longrightarrow S_{\Gamma,G,M,K}$ of the form $\Gamma'h'M' \mapsto \Gamma\phi(h')gM$ for some morphism $\phi : \mathbf{G}' \longrightarrow \mathbf{G}$ of semi-simple linear algebraic $\mathbb{Q}$-groups and some element $g \in G$ such that $\phi(\Gamma') \subset \Gamma$, $\phi(M') \subset gMg^{-1}$, $\phi(K') \subset gKg^{-1}$, and for which the Cartan involution $\theta_{gKg^{-1}}$ preserves $\phi(\mathfrak{g}')$.

The correct form of [2, Theorem 1.1] is therefore:

Theorem 1.2. *Let $\mathbf{G}$ be a connected linear semi-simple algebraic $\mathbb{Q}$-group, $\Gamma \subset \mathbf{G}(\mathbb{Q})^+$ a torsion-free arithmetic lattice of $G := \mathbf{G}(\mathbb{R})^+$, and $M \subset G$ a connected compact subgroup.*

1. *For each choice $K \supset M$ of maximal compact, the arithmetic quotient $S_{\Gamma,G,M} := \Gamma\backslash G/M$ admits a natural structure of $\mathbb{R}_{\mathrm{alg}}$-definable manifold which we call $S_{\Gamma,G,M,K}$, characterized by the following property. Let G/M be endowed with its natural semi-algebraic structure and let $\mathfrak{S} \subset G/M$ be a semialgebraic Siegel set associated to some $\mathbb{Q}$-parabolic and K. Then*

$$\pi_{|\mathfrak{S}} : \mathfrak{S} \longrightarrow S_{\Gamma,G,M,K}$$

is $\mathbb{R}_{\mathrm{alg}}$-definable.

In particular, there exists a semi-algebraic fundamental set $\mathcal{F} \subset G/M$ for the action of Γ on G/M such that

$$\pi_{|\mathcal{F}} : \mathcal{F} \longrightarrow S_{\Gamma,G,M,K}$$

is $\mathbb{R}_{\mathrm{alg}}$-definable.

The $\mathbb{R}_{\mathrm{an}}$-definable manifold $S_{\Gamma,G,M,K}$ extending its $\mathbb{R}_{\mathrm{alg}}$-structure is the one induced by the real-analytic structure with corners of its Borel–Serre compactification[1] $\overline{S_{\Gamma,G,M,K}}^{BS}$.

(2) *Any morphism $S_{\Gamma',G',M',K'} \longrightarrow S_{\Gamma,G,M,K}$ of arithmetic quotients with a choice of maximal compact is $\mathbb{R}_{\mathrm{alg}}$-definable. In particular the Hecke correspondences on $S_{\Gamma,G,M,K}$ are $\mathbb{R}_{\mathrm{alg}}$-definable.*

Proof. The proof of part (1) in [2, §3.1] is correct, so it remains to prove part (2). As in [2] we reduce to $\phi : \mathbf{G'} \to \mathbf{G}$ injective. The key result needed is that for a reductive subgroup $\mathbf{H} \subset \mathbf{G}$, maximal compacts K_H, K_G of H, G, and Siegel set $\mathfrak{S}_H$ of $\mathbf{H}$ associated to K_H, there is a finite set $C \subset \mathbf{G}(\mathbf{Q})$ and a Siegel set $\mathfrak{S}_G$ of $\mathbf{G}$ associated to K_G such that $\mathfrak{S}_H \subset C \cdot \mathfrak{S}_G$. By [4, Theorem 1] (see also Remark 2 therein), this is the case provided $K_H \subset K_G$ and θ_G stabilizes $\mathfrak{g}' \subset \mathfrak{g}$. □

1.4. **Hodge manifolds.** We now show that in the case of pure polarized Hodge manifolds the natural choice of definable structure (coming from the natural choice of maximal compact) is functorial. In the notation of [2], given the restricted Deligne torus $t : \mathbf{S}^1 \to \mathbf{GL}(V_{\mathbb{R}})$ of a pure polarizable $\mathbb{Q}$-Hodge structure on $V_{\mathbb{Q}}$, let $\mathbf{G}$ be the $\mathbb{Q}$-Zariski closure of $t(\mathbf{S}^1)$. Any component of a $\mathbf{G}(\mathbb{R})$-conjugacy class of t is naturally identified with G/M for G a component of $\mathbf{G}(\mathbb{R})$ and M a connected compact subgroup. For any choice of arithmetic lattice $\Gamma \subset \mathbf{G}(\mathbb{Q}) \cap G$ we can associate an arithmetic quotient $S_{\Gamma,t} := \Gamma\backslash G/M$ which parametrizes pure polarizable Hodge structures with the same Hodge numbers as V and whose Mumford–Tate group is contained in $\mathbf{G}$. We call $S_{\Gamma,t}$ a (pure) Hodge manifold. The group G comes equipped with a natural choice of maximal compact subgroup, namely the unitary group K_t for the Hodge metric of the pure Hodge structure on $\mathfrak{g}$ induced by t. Moreover, the Cartan involution of $\mathfrak{g}$ is naturally given by the Weil operator for this Hodge structure. We equip $S_{\Gamma,t}$ with the $\mathbb{R}_{\mathrm{alg}}$-definable structure coming from this choice.

By a morphism of pure Hodge manifolds we mean a morphism $S_{\Gamma',t'} \to S_{\Gamma,t}$ of the form $\Gamma' h' M \mapsto \Gamma\phi(h)gM$ with the above identifications arising from a map of $\mathbb{Q}$-groups $\phi : \mathbf{G'} \to \mathbf{G}$ such that $\phi(\Gamma') \subset \Gamma$ and $\phi \circ t' = gtg^{-1}$. As the induced map $\mathfrak{g}' \to \mathfrak{g}$ is a morphism of $\mathbb{Q}$-Hodge structures, such a map sends $K_{t'}$ to gK_tg^{-1}, and the Cartan involution of gK_tg^{-1} obviously stabilizes $\phi(\mathfrak{g}')$. Thus, we have the following:

Corollary 1.3. *Any morphism $S_{\Gamma',t'} \to S_{\Gamma,t}$ of pure Hodge manifolds equipped with their canonical $\mathbb{R}_{\mathrm{alg}}$-definable structure is $\mathbb{R}_{\mathrm{alg}}$-definable.*

1.5. **Period maps.** The proof of [2, Theorem 1.3] manifestly proves definability with respect to the correct choice of maximal compact. The proof is correct, except that the use of [3, 7.5] in [2, §4.5] to show that inverse images of Siegel sets under the map $\iota : D \to X$ are contained in finitely many Siegel sets should be replaced with [1, 28.1], as the former is for real Siegel sets. The remaining theorems stated in the introduction and their proofs are correct as written.

1.6. **Examples of what can go wrong.**

[1]We use the notation $\overline{S_{\Gamma,G,M,K}}^{BS}$ for the Borel–Serre compactification associated to the choice of maximal compact K. Note that the compactification does not depend on K in the symmetric space case.

1.6.1. *Distinct structures on G/M.* Let $G = \mathbf{SL}_2, \Gamma = \mathbf{SL}_2(\mathbb{Z})$ and $\mathbf{P}$ be the upper triangular subgroup. Let K_x be the maximal compact corresponding to a given $x \in \mathbb{H}$. We claim that a Siegel set $B_N B_A K_x$ is not contained in the union of finitely many Siegel sets $\gamma B'_N B'_A K_y$, for $x \neq y \in \mathbb{H}$. Suppose it were. Then as the latter sets are right K_y invariant, it follows that $S := B_N B_A K_x K_y$ is also contained in finitely many such Siegel sets. We take $y = i$. Then S is the pullback of a set $\pi(S) = B_N B_A K_x i \subset \mathbb{H}$ for the usual identification $G/K_i \cong \mathbb{H}$. Now $K_x i$ is a circle, and B_A scales this circle by real numbers bounded below, and so $\pi(S)$ has unbounded x-co-ordinate, and thus not contained in finitely many K_i-Siegel sets. This proves the claim.

As a corollary, it follows that the induced definable structures on $\Gamma\backslash G$ are distinct, since the fiber product of the two Siegel sets over the quotient has infinitely many connected components, and therefore cannot be definable in an o-minimal structure.

1.6.2. *Morphisms incompatible with definable structures.* The examples in [4, §§B,C] give pairs of morphisms between arithmetic quotients which do not satisfy the compatibility condition on the Cartan involution. As a result, they show that the image of a Siegel set is not contained in finitely many translates of Siegel sets. By the same reasoning as in the previous subsubsection, the morphism is therefore not definable.

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References

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