

LECTURES ON THE B-SEMIAMPLeness CONJECTURE

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ABSTRACT. These lecture notes provide a didactic and gentle exposition of the proof of the b-semiampleness conjecture, and the construction of compact moduli spaces of Calabi–Yau varieties. This includes a deliberately biased introduction to o-minimality and its application in Hodge theory. These notes are based on lecture series given by the authors at the conference *2026 Higher Dimensional Algebraic Geometry*, Taiwan, 7–10 April 2026, at the workshop *Complex Geometry in Bordeaux*, 26–29 May 2026, and at the *Workshop on the Interactions of Hodge Theory and Birational Geometry*, Bochum, 15–16 September 2026.

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1. INTRODUCTION

Semiampleness criteria are fundamental but subtle results in birational geometry. The starting point of the minimal model program is the base point free theorem, and its ultimate goal, the abundance conjecture, is again a semiampleness question. In these lecture notes, we give a gentle introduction to a semiampleness criterion for line bundles arising from Hodge theory; see Theorem 5.1. This is applied to construct projective compactifications of images of period maps and moduli space of Calabi–Yau varieties in §7. The result completes a forty-year program initiated by Griffiths, and simultaneously resolves a long-standing conjecture in birational geometry, the b-semiampleness conjecture, formulated by Kawamata, Shokurov and Prokhorov. For a short account on the history of these problems, we refer to the introduction of [8]. The proof is also interesting from a methodological perspective: it marks the first time that techniques from o-minimal geometry have been successfully applied in birational geometry. In fact, the proof of the semiampleness criterion relies crucially on o-minimal GAGA and definable Artin’s theorem. The reader may safely take o-minimality as a black box; nevertheless, we include a biased introduction in §2, with the aim of illustrating to the algebraic geometer that o-minimality is a rather down-to-earth and elegant piece of real geometry, with application in Hodge theory (cf. §3) and birational geometry.

2. A QUICK TOUR OF O-MINIMALITY AND O-MINIMAL GAGA

We first give a very abbreviated account of o-minimality and o-minimal GAGA.

2.1. Analytification and GAGA. For any complex algebraic space X we may consider its analytification X^{an} . For an affine variety $X = \text{Spec } A$ this is easy to construct: writing $A = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_m)$, we have a closed embedding $X \subset \mathbb{A}^n$, and we consider the analytic space cut out by the polynomials $f_1, \dots, f_m$. In general, X^{an} is the analytic space whose underlying topological space is $X(\mathbb{C})$ with the euclidean topology and whose sheaf of analytic functions is given on an affine étale cover as in the previous example. We can also analytify a coherent sheaf F on X : if $X = \text{Spec } A$ is affine and F has a presentation

$$\mathcal{O}_X^m \xrightarrow{M} \mathcal{O}_X^n \rightarrow F \rightarrow 0$$

for a matrix of algebraic functions M , we just take F^{an} to be the cokernel of $\mathcal{O}_{X^{\text{an}}}^m \xrightarrow{M^{\text{an}}} \mathcal{O}_{X^{\text{an}}}^n$, where M is the same matrix but now considered as a matrix of analytic functions, and in general these local analytifications canonically glue.

Analytic constructions are useful in algebraic geometry because the analytic topology is much finer than the Zariski topology and there are more analytic functions. This applies in particular to Hodge theory, which is fundamentally built on the isomorphism between algebraic de Rham cohomology and singular cohomology via integration, which is intrinsically transcendental.

To come back to the algebraic category we must therefore rely on algebraization theorems which tell us that analytic constructions which glue up globally are in fact algebraic. For *proper* algebraic varieties X , we have a fairly robust package of algebraization theorems:

Theorem 2.1 (Serre’s GAGA¹ [26]). *Let Y be a proper algebraic space. Then the analytification functor*

$$(-)^{\text{an}}: \text{Coh}(X) \rightarrow \text{Coh}(X^{\text{an}})$$

is an equivalence of categories.

Applying this to the ideal sheaf we obtain:

Corollary 2.2 (Chow’s theorem plus). *Let X be a proper algebraic space and $\mathcal{Y} \subset X^{\text{an}}$ a (closed) analytic subspace. Then $\mathcal{Y} = Y^{\text{an}}$ is algebraic.*

We call this “Chow’s theorem plus” because Chow’s theorem, which was proven before Serre’s GAGA, is the case when $\mathcal{Y}$ is reduced (and Y is projective). Applying this to the graph we obtain:

Corollary 2.3. *For X, Y proper algebraic spaces, any morphism $\varphi: X^{\text{an}} \rightarrow Y^{\text{an}}$ is algebraic.*

Corollary 2.4. *A compact analytic space $\mathcal{X}$ has at most one algebraic structure.*

Theorem 2.5 (Artin [3]). *Let X be a proper algebraic space, $\mathcal{V}$ an analytic space, and $\varphi: X^{\text{an}} \rightarrow \mathcal{V}$ a morphism. Then there is a factorization:*

$$\begin{array}{ccc} X^{\text{an}} & \xrightarrow{\varphi} & \mathcal{V} \\ & \searrow f^{\text{an}} & \nearrow \iota \\ & Y^{\text{an}} & \end{array}$$

where $f: X \rightarrow Y$ is “scheme-theoretically” surjective² and ι is a closed embedding.

¹Technically Serre proved this theorem for projective X , and was later upgraded to algebraic spaces. Likewise for Chow’s theorem.

²Meaning $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is injective.

Artin's theorem is in some sense dual to Chow's theorem: while the latter says an analytic subspace of a proper algebraic space is algebraic, the former says an analytic image of a proper algebraic space is algebraic. Artin's theorem is much harder, however, since a subspace of X is described by a coherent sheaf while an image is not.

2.2. GAGA fails for non-proper varieties. If X is not proper, the above algebraization theorems fail immediately.

2.2.1. GAGA fails.

- $(-)^{\text{an}}$ is not essentially surjective. Indeed, if $X = \mathbb{A}^2 \setminus 0$, then X^{an} has *many* line bundles, but the only line bundle on X is the trivial bundle.
- $(-)^{\text{an}}$ is not full. For an affine curve X , any line bundle on X^{an} is trivial, so any two line bundles on X are analytically isomorphic. If the genus of X is positive, however, there are many line bundles on X .

2.2.2. Chow's theorem fails.

- The graph of $e^x: \mathbb{C} \rightarrow \mathbb{C}$ is an analytic subspace of $(\mathbb{A}^2)^{\text{an}}$ which is not algebraic. This is also a non-algebraic morphism $(\mathbb{A}^1)^{\text{an}} \rightarrow (\mathbb{A}^1)^{\text{an}}$.
- We can even make examples of non-algebraic thickenings of algebraic subvarieties. For example, take the subspace cut out by the ideal $(x + e^z y) + (x, y)^2$ in $(\mathbb{A}^3)^{\text{an}}$.
- $(\mathbb{C}^*)^2$ has many non-equivalent algebraic structures. Obviously $\mathbb{G}_m^2$ is one, but so is $\mathbb{P}(V) \setminus s_\infty(E)$ for E an elliptic curve, V the unique (up to scale) nontrivial extension of $\mathcal{O}_E$ by $\mathcal{O}_E$, and s_∞ the section corresponding to the quotient $V \rightarrow \mathcal{O}_E$.

2.2.3. Artin's theorem fails. We may take a smooth projective family $f: X \rightarrow T$ of K3 surfaces where infinitely many fibers contain smooth rational curves which do not deform sideways. Any infinite closed set of such curves may be contracted to yield $g: X^{\text{an}} \rightarrow \mathcal{V}$ which is not algebraic. [6, Ex. 4.13] gives an example where g is finite, but in addition the equivalence relation $X^{\text{an}} \times_{\mathcal{V}} X^{\text{an}}$ is algebraic.

All of these theorems are in some way connected to the appearance of essential singularities at infinity. So if we could somehow build a geometric category of analytic spaces where defining functions and morphisms are required to be “tame” at infinity, we can prevent essential singularities from arising and maybe these statements can be rescued.

Goal 2.6. Come up with a notion of “tame” analytic function and “tame” subset of $\mathbb{C}^n$, and build a category of spaces which are locally zero sets of tame analytic functions on tame open sets, glued by tame transition functions.

2.3. o-minimal structures. An o-minimal structure specifies a notion of “tame” subset of real euclidean space, and a notion of “tame” maps between them, that will allow for quite complicated constructions while preserving important finiteness properties.

Definition 2.7. A structure S is a collection $(S_n)_{n \in \mathbb{N}}$ where each S_n is a set of subsets of $\mathbb{R}^n$ satisfying the following conditions:

- (1) Each S_n is closed under finite intersections, finite unions, and complements;
- (2) The collection S is closed under finite Cartesian products and coordinate projection;
- (3) For every polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$, the zero set

$$(P = 0) := \{x \in \mathbb{R}^n \mid P(x) = 0\} \subset \mathbb{R}^n$$

is an element of S_n .

We refer to an element $U \in S_n$ as an S -definable subset $U \subset \mathbb{R}^n$. We say a map $f: U \rightarrow V$ of S -definable sets is S -definable if the graph is. When the structure S is clear from context, we will often just refer to “definable” sets and functions.

The definable sets/functions should be thought of as the sets/functions that are “constructible” within the theory. Its easy to see, for example, that images and inverse images of definable sets under definable maps are definable, and that compositions of definable functions are definable.

In general, the operations in the axioms say that definability is preserved under constructions using only first order statements, and this is a useful way of thinking about it. For example, while infinite unions or intersections of definable subsets are not definable, many topological constructions with respect to the euclidean topology are definable:

Proposition 2.8. *Let S be a structure, and endow $\mathbb{R}^n$ with the euclidean topology. Closures, interiors, and boundaries of definable sets are definable.*

Proof. We just show that the closure of a definable set $U \subset \mathbb{R}^n$ is defined by a first order formula and leave the rest as an exercise:

$$\overline{U} = \left\{ x \in \mathbb{R}^n \mid \forall \epsilon > 0, \exists y \in U \text{ s.t. } \sum_i (x_i - y_i)^2 < \epsilon \right\}$$

□

Now lets see some examples of structures.

Example 2.9. The collection S of constructible real algebraic sets—that is, $S_n =$ the Boolean algebra generated by sets of the form $(P = 0)$ for $P \in \mathbb{R}[x_1, \dots, x_n]$ —is *not* a structure. Indeed, for any $P \in \mathbb{R}[x_1, \dots, x_n]$, the image of the projection of $(x_0^2 - P = 0)$ forgetting x_0 is $(P \geq 0)$.

Example 2.10. Let $\mathbb{R}_{\text{alg}}$ be the collection of real semialgebraic subsets of $\mathbb{R}^n$ —that is, $(\mathbb{R}_{\text{alg}})_n$ is the Boolean algebra generated by sets of the form $(P \geq 0)$ for $P \in \mathbb{R}[x_1, \dots, x_n]$. Then $\mathbb{R}_{\text{alg}}$ is a structure. By the Tarski–Seidenberg theorem (see for example [30, Ch. 2]), coordinate projections of real semialgebraic sets are real semialgebraic, and the other axioms are easy to verify. $\mathbb{R}_{\text{alg}}$ is therefore a structure, in fact the structure generated by the real algebraic sets given Example 2.9—that is, the structure whose definable sets are defined by first order formulas involving real algebraic sets.

In particular, the $\mathbb{R}_{\text{alg}}$ -definable subsets of $\mathbb{R}$ are just finite unions of intervals (where we allow open, closed, half-open, and length 0 and infinite intervals).

So far we have not imposed any tameness conditions; the following at first sight seems quite weak, but in fact because of the projection axiom is very strong.

Definition 2.11. A structure S is said to be *o-minimal* if $S_1 = (\mathbb{R}_{\text{alg}})_1$ —that is, if the S -definable subsets of the real line are exactly finite unions of intervals.

Of course this means $\mathbb{R}_{\text{alg}}$ is o-minimal.

Example 2.12. Let $\mathbb{R}_{\text{exp}}$ be the structure generated by $\mathbb{R}_{\text{alg}}$ and the graph of the real exponential $\exp: \mathbb{R} \rightarrow \mathbb{R}$. This means definable subsets are defined by first order formulas involving polynomials and the real exponential. Then $\mathbb{R}_{\text{exp}}$ is o-minimal by a result of Wilkie [32].

$\mathbb{R}_{\text{exp}}$ contains some very complicated sets, for example:

$$\left\{ (y, z) \in \mathbb{R}^2 \mid \exists x \in (0, 1), \quad e^{x^2} + e^{xy} + e^{(1-x)z} < 4 \right\}.$$

Nonetheless, any projection of this set to the real line is a finite union of intervals.

Example 2.13. Let $\mathbb{R}_{\sin}$ be the structure generated by $\mathbb{R}_{\text{alg}}$ and the graph of $\sin: \mathbb{R} \rightarrow \mathbb{R}$. $\mathbb{R}_{\sin}$ is not o-minimal as $\pi\mathbb{Z} = \sin^{-1}(0)$ is definable and infinite discrete.

Example 2.14. Let $\mathbb{R}_{\text{an}}$ be the structure generated by $\mathbb{R}_{\text{alg}}$ and the graphs of *all* restrictions $f|_{B(R-\epsilon)}$ of real analytic functions $f: B(R) \rightarrow \mathbb{R}$ on a finite radius open euclidean ball (centered at the origin) to a strictly smaller radius ball. Via the embedding $\mathbb{R}^n \subset \mathbb{RP}^n$, this is equivalent to the structure of subsets of $\mathbb{R}^n$ that are subanalytic in $\mathbb{RP}^n$. $\mathbb{R}_{\text{an}}$ is o-minimal by van den Dries [30], using Gabrielov’s theorem of the complement. Note that while $\sin(x)$ is not $\mathbb{R}_{\text{an}}$ -definable, its restriction to any finite interval is.

Example 2.15. Let $\mathbb{R}_{\text{an,exp}}$ be the structure generated by $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$. $\mathbb{R}_{\text{an,exp}}$ is o-minimal by a result of van den Dries–Miller [31]. All of the Hodge-theoretic constructions we discuss will use $\mathbb{R}_{\text{an,exp}}$.

A definable topological space informally consists of finitely many definable subsets of $\mathbb{R}^n$ glued via continuous definable transitions maps. Luckily, for us, all of the definable topological spaces we will need (namely the ones whose underlying topological space is regular) are embeddable in $\mathbb{R}^n$.

Definition 2.16. For a definable subset $X \subset \mathbb{R}^n$, we equip it with its subspace topology, and define the associated definable site $\underline{X}$ to be the site consisting of definable open subsets $U \subset X$ and whose covers are finite covers by definable open subsets. $\underline{X}$ comes equipped with a sheaf of continuous definable $\mathbb{R}$ -valued functions $\mathcal{C}_X^0$, and we obtain a locally ringed site $(\underline{X}, \mathcal{C}_X^0)$ (over $\mathbb{R}$). A locally ringed site of this form will be called a regular definable topological space.

Importantly, with respect to an o-minimal structure, regular definable topological spaces $X = (\underline{X}, \mathcal{C}_X^0)$ have many nice finiteness properties:

- Any regular definable topological space X has finitely many components.
- Any regular definable topological space X has a finitely definable triangulation. It follows that X has a finite definable cover by contractible definable open sets.
- For any morphism $f: X \rightarrow Y$ of regular definable topological spaces with discrete fibers, $|f^{-1}(y)|$ is finite and uniformly bounded.

Note that it is essential in the sheaf theory of S that we only ever allow finite covers by definable open subsets, or else we would lose all the finiteness properties we’re trying so hard to preserve!

2.4. o-minimality and complex geometry. While o-minimal structures are about $\mathbb{R}^n$, we can start using them in complex analysis using the identification $\mathbb{C}^n = \mathbb{R}^{2n}$ by taking real and imaginary parts. The resulting theory was pioneered by Peterzil–Starchenko. They proved the following, which is evidence we have made a step towards achieving Goal 2.6:

Theorem 2.17 (Definable Chow, [24]). *Let $\mathcal{Y} \subset (\mathbb{A}^n)^{\text{an}} = \mathbb{C}^n$ be a reduced analytic subspace whose underlying set is definable in an o-minimal structure. Then $\mathcal{Y} = Y^{\text{an}}$ is algebraic.*

Example 2.18. As a baby case of Theorem 2.17, let’s observe that an entire function $f: \mathbb{C} \rightarrow \mathbb{C}$ which is definable in an o-minimal structure is algebraic by Casorati–Weierstrass, which in particular says that if f had an essential singularity it would have infinite fibers.

Example 2.19. As a slightly older baby case of Theorem 2.17, consider the $n = 2$ case where $\mathcal{Y}$ is purely 1-dimensional. One can show that there is a projection $\pi_1: \mathbb{C}^2 = \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ whose restriction to $\mathcal{Y}$ is finite of degree d . Then the function

$$f: \mathbb{C} \rightarrow \text{Sym}^d \mathbb{C}, \quad x \mapsto \pi_1^{-1}(x) \cap \mathcal{Y}$$

is holomorphic and definable, where we take the fiber with multiplicity. Now consider the function

$$P = \prod_{t \in f(z)} (w - t).$$

Its coefficients (in w) are the elementary symmetric polynomials applied to f , hence definable entire functions, hence polynomials. Thus, P is a polynomial cutting out $\mathcal{Y}$.

From now on let's fix an o-minimal structure S ; in the following by “definable” we always mean definable in S .

Definition 2.20. A basic definable analytic space consists of:

- An open definable $U \subset \mathbb{C}^n$;
- An analytic subspace $X = V(f_1, \dots, f_m) \subset U$ cut out by definable holomorphic functions $f_1, \dots, f_m: U \rightarrow \mathbb{C}$;
- The sheaf $\mathcal{O}_X := \mathcal{O}_U / (f_1, \dots, f_m) \mathcal{O}_U$ considered on the definable site $\underline{X}$. Here the sheaf $\mathcal{O}_U$ is the sheaf of definable holomorphic functions on the definable site $\underline{U}$.

Informally, a definable analytic space is formed by gluing *finitely* many basic definable analytic spaces by definable analytic transition functions. Slightly more formally, it is a locally ringed site $(\underline{\mathfrak{X}}, \mathcal{O}_X)$ (over $\mathbb{C}$), where $\underline{\mathfrak{X}}$ is the definable site of a regular definable topological space $\mathfrak{X}$, which comes from a basic definable analytic space after passing to a cover.

2.5. o-minimal GAGA. Fix an o-minimal structure. Then the category of definable analytic spaces achieves Goal 2.6. Indeed, echoing the construction in the beginning of Section 2.1, we have “definabilization” functors which fit into the diagrams

$$\begin{array}{ccc} (\text{Algebraic spaces}/\mathbb{C}) & \xrightarrow{(-)^{\text{def}}} & (\text{Definable analytic spaces}) \\ & \searrow^{(-)^{\text{an}}} \quad \swarrow_{(-)^{\text{an}}} & \\ & (\text{Analytic spaces}) & \end{array}$$

and, for any algebraic space X ,

$$\begin{array}{ccc} \text{Coh}(X) & \xrightarrow{(-)^{\text{def}}} & \text{Coh}(X^{\text{def}}) \\ & \searrow^{(-)^{\text{an}}} \quad \swarrow_{(-)^{\text{an}}} & \\ & \text{Coh}(X^{\text{an}}) & \end{array}$$

It is shown in [6] that the coherent sheaf theory of X^{def} has all the nice properties you would expect (in particular, a definable Oka coherence theorem and a definable Nullstellensatz), and we have:

Theorem 2.21 (Definable GAGA, [6]). *Let X be an algebraic space. Then the definabilization functor*

$$(-)^{\text{def}}: \text{Coh}(X) \rightarrow \text{Coh}(X^{\text{def}})$$

is fully faithful, exact, and its essential image is closed under subobjects and quotient objects.

Note that definabilization does *not* give an equivalence of categories—it is not essentially surjective. See [6]. Nonetheless, we still deduce by the same proof:

Corollary 2.22 (Definable Chow’s theorem plus, [6]). *Let X be an algebraic space and $\mathcal{Y} \subset X^{\text{def}}$ a (closed) definable analytic subspace. Then $\mathcal{Y} = Y^{\text{def}}$ is algebraic.*

Note that a *reduced* definable analytic subspace of a definable analytic space is the same thing as a reduced analytic subspace whose underlying set is definable (this is not obvious! [6, Prop. 2.45]).

Corollary 2.23. *For X, Y algebraic spaces, any morphism $\varphi: X^{\text{def}} \rightarrow Y^{\text{def}}$ is algebraic.*

Corollary 2.24. *A definable analytic space $\mathcal{X}$ has at most one algebraic structure.*

Theorem 2.25 (Definable Artin, [6]). *Let X be an algebraic space, $\mathcal{V}$ a definable analytic space, and $\varphi: X^{\text{def}} \rightarrow \mathcal{V}$ a proper morphism. Then there is a factorization:*

$$\begin{array}{ccc} X^{\text{def}} & \xrightarrow{\varphi} & \mathcal{V} \\ & \searrow f^{\text{def}} \quad \nearrow \iota & \\ & Y^{\text{def}} & \end{array}$$

where $f: X \rightarrow Y$ is “scheme-theoretically” surjective and ι is a closed embedding.

The following is not strictly speaking necessary, but will clarify the proofs of the semi-ampleness results.

Theorem 2.26 (Definable Stein factorization, [7]). *Let X be seminormal definable analytic space and $f: X \rightarrow Z$ a morphism of definable analytic spaces whose fibers are compact. Then there is a definable analytic Stein factorization, that is, a diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ & \searrow g \quad \nearrow h & \\ & Y & \end{array}$$

that analytifies to the Stein factorization in the analytic category. In particular, g has connected fibers, $\mathcal{O}_Y \rightarrow g_*\mathcal{O}_X$ is an isomorphism, and h is quasifinite.

Note that cohomology is *not* well-behaved in the definable analytic category, so the usual Grauert proof on the coherence of higher push-forwards along proper maps does *not* apply. Indeed, an example of Esnault–Kerz [15] shows that push-forward along proper definable analytic morphisms does not preserve coherence, and so Stein factorization does not hold for X non-reduced.

3. O-MINIMALITY IN HODGE THEORY

3.1. Definable Riemann–Hilbert. In this section we justify the sense in which “Hodge theory is definable in $\mathbb{R}_{\text{an}, \text{exp}}$ ”.

Let’s first quickly review the Riemann–Hilbert correspondence.

Lemma 3.1 (Deligne [11]). *Fix a (set-theoretic) section $s: \mathbb{C}^* \rightarrow \mathbb{C}$ of $e(-) := e^{2\pi i(-)}: \mathbb{C} \rightarrow \mathbb{C}^*$. Let $(\mathcal{X}, \mathcal{D})$ be a log smooth analytic variety and $(\mathcal{V}, \nabla)$ a flat vector bundle on $\mathcal{X} \setminus \mathcal{D}$. Then $(\mathcal{V}, \nabla)$ admits a unique logarithmic flat vector bundle extension $(\bar{\mathcal{V}}, \bar{\nabla})$ to $(\mathcal{X}, \mathcal{D})$ for which the eigenvalues of the residue are contained in $\text{im}(s)$. In particular, if $(\mathcal{V}, \nabla)$ has unipotent local monodromy, then it admits a unique logarithmic flat vector bundle extension with nilpotent residue, and this extension is functorial with respect to pullback.*

Now, we claim the functor

$$(3.1) \quad (\text{Flat vector bundles on } X \text{ with regular singularities}) \rightarrow (\mathbb{C}\text{-local systems on } X^{\text{an}})$$

sending (V, ∇) to the flat sections $(V^{\text{an}})^{\nabla}$ of the analytification is an equivalence. We construct its inverse. Choose a log smooth compactification $(\bar{X}, D)$ of X . For any complex

local system $V_{\mathbb{C}}$, we form the analytic flat vector bundle $(V_{\mathcal{O}^{\text{an}}} := \mathcal{O}_{X^{\text{an}}} \otimes_{\mathbb{C}_{X^{\text{an}}}} V_{\mathbb{C}}, d \otimes 1)$, which by the lemma admits an extension to a logarithmic flat vector bundle $(\bar{V}_{\mathcal{O}^{\text{an}}}, \bar{\nabla})$ on $\bar{X}$. By ordinary GAGA, this flat vector bundle has a unique algebraic structure $(\bar{V}_{\mathcal{O}}, \bar{\nabla})$, which restrict to an algebraic structure $(V_{\mathcal{O}}, \nabla)$ of $(V_{\mathcal{O}^{\text{an}}}, \nabla)$. This construction is functorial, and is easily checked to give an inverse.

Example 3.2. Let $f: Y \rightarrow X$ be a smooth proper morphism of algebraic spaces. Let's assume X is smooth, and that f admits a semistable model $\bar{f}: \bar{Y} \rightarrow \bar{X}$, where $\bar{Y}$ and $\bar{X}$ are log smooth compactifications of Y and X , respectively. Then the algebraic structure on $V_{\mathbb{C}} = R^k f_*^{\text{an}} \mathbb{C}_{Y^{\text{an}}}$ is given by algebraic de Rham cohomology $(R^k f_* \Omega_{Y/X}^{\bullet}, \nabla)$, where ∇ is the Gauss–Manin connection. Indeed, by a theorem of Grothendieck,

$$(R^k f_* \Omega_{Y/X}^{\bullet}, \nabla)^{\text{an}} \cong (R^k f_*^{\text{an}} \Omega_{Y^{\text{an}}/X^{\text{an}}}^{\bullet}, \nabla)$$

via the natural map, while

$$R^k f_*^{\text{an}} \mathbb{C}_{Y^{\text{an}}} \cong (R^k f_*^{\text{an}} \Omega_{Y^{\text{an}}/X^{\text{an}}}^{\bullet})^{\nabla}$$

since the natural morphism $(f^{\text{an}})^{-1} \mathcal{O}_{Y^{\text{an}}} \rightarrow \Omega_{Y^{\text{an}}/X^{\text{an}}}^{\bullet}$ is a quasi-isomorphism. Finally, $(R^k \bar{f}_* \Omega_{\bar{Y}/\bar{X}}^{\bullet}, \nabla)$ provides the required logarithmic extension with nilpotent residues.

Now, for a local system $V_{\mathbb{C}}$ on X^{an} , we obtain two structures on the analytic flat vector bundle $(V_{\mathcal{O}^{\text{an}}}, \nabla)$ as a flat vector bundle on X^{def} :

- First, we may form $(\mathcal{O}_{X^{\text{def}}} \otimes_{\mathbb{C}_{X^{\text{def}}}} V_{\mathbb{C}}, d \otimes 1)$, which makes sense because X admits a definable cover by contractible open sets. A local frame for this structure is just a flat frame.
- We may also form $(V_{\mathcal{O}^{\text{def}}}, \nabla) := (V_{\mathcal{O}}, \nabla)^{\text{def}}$, where $(V_{\mathcal{O}}, \nabla)$ is the algebraic flat vector bundle given by the Riemann–Hilbert correspondence (3.1). Here, the local frames are the algebraic frames.

These are not in general isomorphic [10], but for the local systems that arise in Hodge theory they are:

Theorem 3.3 ([10]). *Take $S = \mathbb{R}_{\text{an}, \text{exp}}$. Suppose the local monodromy of $V_{\mathbb{C}}$ has norm one eigenvalues with respect to some (hence any) algebraic compactification. Then there is a functorial isomorphism*

$$(V_{\mathcal{O}}, \nabla)^{\text{def}} \cong (\mathcal{O}_{X^{\text{def}}} \otimes_{\mathbb{C}_{X^{\text{def}}}} V_{\mathbb{C}}, d \otimes 1).$$

Sketch of proof. Concretely, we must show that for any algebraic flat vector bundle (with regular singularities) satisfying the local monodromy condition, there is a definable cover for which the change of basis between an algebraic frame and a flat frame is a matrix of definable analytic functions.

For simplicity, assume X is smooth, and let $X \subset \bar{X}$ be a log smooth compactification. We may cover $\bar{X}$ with finitely many polydisks, and since all restricted analytic functions are definable, after shrinking each polydisk we obtain a definable cover.

Let $U = (\Delta^*)^k \times \Delta^\ell$ be one of these polydisks with coordinates q_i . By definition of the algebraic structure $(V_{\mathcal{O}}, \nabla)$, it admits a logarithmic extension $(\bar{V}_{\mathcal{O}}, \nabla)$ the eigenvalues of whose residues are contained in $\text{im}(s)$. On the other hand, we can explicitly make the logarithmic extension of $(\mathcal{O}_{X^{\text{an}}} \otimes_{\mathbb{C}_{X^{\text{an}}}} V_{\mathbb{C}}, d \otimes 1)$ on the polydisk: if T_i is the monodromy around the i th puncture, then there is a unique logarithm N_i of T_i such that the eigenvalues of $\frac{1}{2\pi i} N_i$ are in $\text{im}(s)$. Then for any multivalued section v of $V_{\mathbb{C}}|_U$, $\exp(\sum_i z_i \otimes N_i)(1 \otimes v)$ is a single-valued section, where $z_i = e^{-1}(q_i) = \frac{1}{2\pi i} \log q_i$. By taking v_j to be a basis at a point, we obtain a trivialization of $(\mathcal{O}_{X^{\text{an}}} \otimes_{\mathbb{C}_{X^{\text{an}}}} V_{\mathbb{C}}, d \otimes 1)$ which therefore defines a (trivial) extension, and the residue of $\bar{\nabla}$ around the i th boundary component is exactly

$\frac{1}{2\pi i}N_i$. According to Lemma 3.1, the change of basis matrix from the algebraic frame of $(\bar{V}_{\mathcal{O}}, \bar{\nabla})$ and the frame $\exp(\sum_i z_i \otimes N_i)(1 \otimes v_j)$ is analytic, hence after shrinking the polydisks it is definable, hence its restriction to U is definable.

Now, further cover each punctured disk with bounded angular sectors, and we obtain a definable cover of each polydisk by contractible opens. It remains to argue that on this cover, the change of basis from the frame $\exp(\sum_i z_i \otimes N_i)(1 \otimes v_j)$ to the frame $1 \otimes v_j$ is definable. But notice that on this cover z_i is definable, since $q_i = e(z_i) = e^{-2\pi y_i}(\cos 2\pi x_i + i \sin 2\pi x_i)$ where $z_i = x_i + iy_i$ and x_i is *bounded*. If $N_i = N_i^{ss} + N_i^n$ is the Jordan decomposition into semisimple and nilpotent parts and we take v_j to be a basis of eigenvectors for N_i^{ss} , then by the monodromy condition the eigenvalue λ_{ij} of $\frac{1}{2\pi i}N_i^{ss}$ on v_j is real. Then, on the one hand $\exp(z_i \otimes N_i^n)$ is a polynomial in z_i , hence definable, while $\exp(z_i \otimes N_i^{ss})$ acts with eigenvalue $e^{-2\pi \lambda_{ij} y_i}(\cos \lambda_{ij} x_i + i \sin \lambda_{ij} x_i)$ on $1 \otimes v_j$, hence is definable. $\square$

3.2. Lightning recap of Hodge theory. A variation of Hodge structures (VHS) is an abstraction from the package we get from Example 3.2—it should be thought of as a kind of linearization of the algebraic family. For a smooth algebraic space X , the most important data associated to a VHS on X is a complex local system $V_{\mathbb{C}}$ and a filtration $F^{\bullet}V_{\mathcal{O}}$ on the associated algebraic flat vector bundle. However, strictly speaking the VHS is only determined by these two pieces if it is polarizable. We take a step back and make all the required definitions to understand what a VHS is and see how its various parts fit together.

Definition 3.4. (1) A pure complex variation of Hodge structures (pure $\mathbb{C}$ -VHS) consists of a triple $V = (V_{\mathbb{C}}, F^{\bullet}V_{\mathcal{O}^{\text{an}}}, F^{\bullet}V_{\bar{\mathcal{O}}^{\text{an}}})$ where:

- $V_{\mathbb{C}}$ is a complex local system
- $F^{\bullet}V_{\mathcal{O}^{\text{an}}}$ is a locally splittable decreasing filtration of the analytic flat vector bundle $V_{\mathcal{O}^{\text{an}}} := \mathcal{O}_{X^{\text{an}}} \otimes_{\mathbb{C}_{X^{\text{an}}}} V_{\mathbb{C}}$
- $F^{\bullet}V_{\bar{\mathcal{O}}^{\text{an}}}$ is a locally splittable filtration of the anti-analytic flat vector bundle $V_{\bar{\mathcal{O}}^{\text{an}}} := \bar{\mathcal{O}}_{X^{\text{an}}} \otimes_{\mathbb{C}_{X^{\text{an}}}} V_{\mathbb{C}}$

These filtrations satisfy:

- $F^{\bullet}V_{\mathcal{O}^{\text{an}}}$ is Griffiths tranverse:

$$\nabla(F^p V_{\mathcal{O}^{\text{an}}}) \subset \Omega_{X^{\text{an}}} \otimes F^{p-1} V_{\mathcal{O}^{\text{an}}}, \text{ for each } p.$$

Likewise for $F^{\bullet}V_{\bar{\mathcal{O}}^{\text{an}}}$.

- There is a splitting

$$V_{C_X^{\infty}} := C_X^{\infty} \otimes_{\mathbb{C}_{X^{\text{an}}}} V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$$

such that the filtration $F^{\bullet} := (F^{\bullet}V_{\mathcal{O}^{\text{an}}})_{C_X^{\infty}}$ (resp. $F'^{\bullet} := (F^{\bullet}V_{\bar{\mathcal{O}}^{\text{an}}})_{C_X^{\infty}}$) on $V_{C_X^{\infty}}$ induced by $F^{\bullet}V_{\mathcal{O}^{\text{an}}}$ (resp. $F^{\bullet}V_{\bar{\mathcal{O}}^{\text{an}}}$) is $F^p = \bigoplus_{s \geq p} V^{s,t}$ (resp. $F'^q = \bigoplus_{t \geq q} V^{s,t}$). The uniquely determined integer w is called the *weight*.

(2) V is *polarizable* if there is a flat hermitian form b on $V_{\mathbb{C}}$ for which:

- The splitting $V_{C_X^{\infty}} = \bigoplus V^{p,q}$ is b -orthogonal.
- $(-1)^p b|_{V^{p,q}}$ is positive definite for all p, q .

Note that in this case $F^{\bullet}V_{\bar{\mathcal{O}}^{\text{an}}}$ is *determined* from $F^{\bullet}V_{\mathcal{O}^{\text{an}}}$ by $F'^q = (F^{w-q+1})^{\perp}$ where the perp is taken with respect to b .

(3) A weight w $\mathbb{R}$ -VHS consists of a real local system $V_{\mathbb{R}}$ and a filtration $F^{\bullet}V_{\mathcal{O}^{\text{an}}}$ on the associated flat bundle for which $(V_{\mathbb{C}}, F^{\bullet}V_{\mathcal{O}^{\text{an}}}, \overline{F^{\bullet}V_{\mathcal{O}^{\text{an}}}})$ is a $\mathbb{C}$ -VHS. Equivalently, this is a real local system $V_{\mathbb{R}}$ with a $\mathbb{C}$ -VHS structure on its complexification $V_{\mathbb{C}}$ for which complex conjugation $V_{\mathbb{C}} \rightarrow V_{\bar{\mathbb{C}}}$ is a morphism of $\mathbb{C}$ -VHS. An

- ($\mathbb{R}$ -)polarization of a weight w $\mathbb{R}$ -VHS V is a $(-1)^w$ -symmetric form q on $V_{\mathbb{R}}$ for which $i^w q(x, \bar{y})$ is a polarization over $\mathbb{C}$. The symmetry/antisymmetry is forced by the alternating sign condition on the complex polarization. We say a pure $\mathbb{R}$ -VHS is polarizable if it admits an ($\mathbb{R}$ -)polarization. A $\mathbb{R}$ -VHS V is polarizable iff the complexification is polarizable: in the nonobvious backward direction, we may take $q(u, v) = (-i)^w b(u, \bar{v}) + i^w b(v, \bar{u})$.
- (4) If $R \subset \mathbb{R}$ is a subring, a pure R -VHS is a R -local system V_R and a pure $\mathbb{R}$ -VHS structure on $V_{\mathbb{R}}$. By an (R -)polarization we mean a form q on V_R which is a $\mathbb{R}$ -polarization over $\mathbb{R}$. We say a pure R -VHS is (R -)polarizable if it admits a (R -)polarization.
 - (5) A mixed $\mathbb{C}$ -VHS is a quadruple $(V_{\mathbb{C}}, W_{\bullet} V_{\mathbb{C}}, F^{\bullet} V_{\mathcal{O}^{\text{an}}}, F^{\bullet} V_{\bar{\mathcal{O}}^{\text{an}}})$ where $W_{\bullet} V_{\mathbb{C}}$ is a descending filtration of $V_{\mathbb{C}}$ by $\mathbb{C}$ -local systems such that $(\text{gr}_w^W V_{\mathbb{C}}, F^{\bullet} \text{gr}_w^W V_{\mathcal{O}^{\text{an}}}, F^{\bullet} \text{gr}_w^W V_{\bar{\mathcal{O}}^{\text{an}}})$ is a weight w $\mathbb{C}$ -VHS for each w . Likewise, for $R \subset \mathbb{R}$, we require $V_{\mathbb{C}}$ to descend to R , $W_{\bullet} V_{\mathbb{C}}$ to descend to the field of fractions K , and $F^{\bullet} V_{\bar{\mathcal{O}}^{\text{an}}} = \overline{F^{\bullet} V_{\mathcal{O}^{\text{an}}}}$. Note that each graded piece has a natural R -structure.
 - (6) Finally, a mixed R -VHS is graded-polarizable if each weight graded piece is graded-polarizable.

Example 3.5. The motivating example is the following. If $f: Y \rightarrow X$ a morphism of quasiprojective varieties for which $R^k f_* \mathbb{Q}_{Y^{\text{eucl}}}$ is a local system.³ Then the torsion-free part of $R^k f_* \mathbb{Z}_{Y^{\text{eucl}}}$ underlies an admissible graded-polarizable mixed $\mathbb{Z}$ -VHS.

Theorem 3.6 (Schmid [25, 13]). *Let $j: (\Delta^*)^n \rightarrow \Delta^n$ be the inclusion, V a polarizable pure $\mathbb{C}$ -VHS on $(\Delta^*)^n$, and $\bar{V}_{\mathcal{O}}$ the Deligne extension with respect to some section of the exponential. Then $F^{\bullet} \bar{V}_{\mathcal{O}^{\text{an}}} := V_{\mathcal{O}} \cap j_* F^{\bullet} V_{\mathcal{O}^{\text{an}}}$ is a locally splittable filtration of $\bar{V}_{\mathcal{O}^{\text{an}}}$.*

Corollary 3.7. *For an algebraic space X and a polarizable pure $\mathbb{C}$ -VHS V on X^{an} , $F^{\bullet} V_{\mathcal{O}^{\text{an}}}$ comes from an algebraic filtration $F^{\bullet} V_{\mathcal{O}}$ of the (canonical) algebraic structure $V_{\mathcal{O}}$ of the flat vector bundle associated to $V_{\mathbb{C}}$. For any log smooth compactification $X \subset \bar{X}$, this filtration extends to a locally splittable filtration of any Deligne extension.*

For this reason, for $R \subset \mathbb{R}$ we will henceforth refer to the data determining a polarizable pure R -VHS as $(V_R, F^{\bullet} V_{\mathcal{O}})$.

Theorem 3.8 (Borel, see [25]). *For any polarizable pure $\mathbb{Z}$ -VHS $(V_{\mathbb{Z}}, F^{\bullet} V_{\mathcal{O}})$, $V_{\mathbb{Z}}$ has quasi-unipotent local monodromy.*

In particular, by Theorem 3.3 we have:

Corollary 3.9. *For any local system $V_{\mathbb{Z}}$ underlying a graded-polarizable mixed $\mathbb{Z}$ -VHS, the flat and algebraic definable structures agree:*

$$(V_{\mathcal{O}}, \nabla)^{\text{def}} \cong (\mathcal{O}_{X^{\text{def}}} \otimes V_{X^{\text{def}}}, d \otimes 1).$$

Theorem 3.10 (Deligne [12]). *For any polarizable $\mathbb{C}$ -VHS $(V_{\mathbb{C}}, F^{\bullet} V_{\mathcal{O}})$, any isotypic component $W \subset V_{\mathbb{C}}$ of the local system underlies a sub- $\mathbb{C}$ -VHS.*

Theorem 3.11 (Theorem of the fixed part [25, 29, 2]). *For an admissible graded-polarizable mixed $\mathbb{C}$ -VHS V , the fixed part $\underline{H^0(V)} \subset V_{\mathbb{C}}$ underlies a sub-VHS.*

Deligne's theorem has the following important corollary:

Corollary 3.12. *Let V be a polarizable pure $\mathbb{Z}$ -VHS on a smooth algebraic variety X . Any rank one sub-local system $W \subset V_{\mathbb{C}}$ has torsion monodromy.*

³This is always the case after base-changing to a dense open Zariski subset of X , e.g., restricting on the locus where f is a topologically locally trivial fibration.

Proof. The isotypic component of V containing W is defined over $\bar{\mathbb{Z}}$. An appropriate wedge of such isotypic component is isomorphic to a suitable power $W^{\otimes r}$, and defined over $\bar{\mathbb{Z}}$. By Deligne's theorem it is also unitary. The same is true for every Galois conjugate of $W^{\otimes r}$, so the monodromies of $W^{\otimes r}$ are algebraic integers all of whose complex embeddings have norm one, hence they are roots of unity by Kronecker's theorem. $\square$

3.3. Algebraicity of Stein-factorized period images. Suppose $(V_{\mathbb{Z}}, F^{\bullet}V_{\mathcal{O}})$ is a polarizable pure $\mathbb{Z}$ -VHS. We denote by $\widetilde{X^{\text{an}}}^{V_{\mathbb{Z}}}$ the minimal covering space of X^{an} on which $V_{\mathbb{Z}}$ is trivialized. Choosing a basepoint x , we obtain a canonical identification $\pi^*V_{\mathbb{Z}} = \underline{V_{\mathbb{Z},x}}$. If we choose a $\mathbb{Z}$ -polarization b , then we obtain a period map

$$(3.2) \quad \widetilde{X^{\text{an}}}^{V_{\mathbb{Z}}} \xrightarrow{\tilde{\varphi}} \check{\mathbb{D}}(V_{\mathbb{C},x})$$

where $\check{\mathbb{D}}(V_{\mathbb{C},x})$ is the variety of b -isotropic filtrations of the relevant shape. Crucially, Corollary 3.9 implies the following:

Key Fact 3.13. *(3.2) is definable in the following sense. For any definable open subset $\mathcal{U} \subset X^{\text{def}}$ admitting a continuous lift $\mathcal{U} \subset \widetilde{X^{\text{an}}}^{V_{\mathbb{Z}}}$, the resulting map $\mathcal{U} \rightarrow \check{\mathbb{D}}(V_{\mathbb{C},x})^{\text{def}}$ is a morphism of definable analytic varieties*

If $\mathbb{D} \subset \check{\mathbb{D}}(V_{\mathbb{C},x})$ is the open set of filtrations coming from b -polarized Hodge structures with the relevant Hodge numbers, and $\Gamma \subset \text{GL}(V_{\mathbb{Z},x})$ is the image of the monodromy representation of $V_{\mathbb{Z}}$, Γ acts properly discontinuously on $\mathbb{D}$, so we can consider

$$(3.3) \quad X^{\text{an}} \xrightarrow{\varphi} \Gamma \backslash \mathbb{D}$$

Theorem 3.14 ([6]). *We have a diagram*

$$\begin{array}{ccc} X^{\text{an}} & \xrightarrow{\varphi} & \Gamma \backslash \mathbb{D} \\ & \searrow f^{\text{an}} & \nearrow \nu \\ & Y^{\text{an}} & \end{array}$$

where f is dominant and ν is finite.

Sketch of proof. We may pass to an étale cover of X by adjoining level structure of $V_{\mathbb{Z}}$, since the desired diagram will then be obtained as a finite group quotient. We may thereby assume $V_{\mathbb{Z}}$ has neat local monodromy.

Step 1. Taking a smooth compactification $\bar{X}$ with snc boundary, a theorem of Griffiths implies: (a) V extends over boundary components with trivial local monodromy; (b) if there are no such boundary components φ is proper. Thus we may assume φ is proper. In particular, taking the Stein factorization of φ , the claimed map $f: X^{\text{an}} \rightarrow \mathcal{Y}$ exists on the level of analytic spaces and additionally has the property that it is proper and $\mathcal{O}_{\mathcal{Y}} \rightarrow f_*\mathcal{O}_{X^{\text{an}}}$ is an isomorphism.

Step 2. Using Key Fact 3.13, there is a definable (hence algebraic, by definable Chow) dense Zariski open $U \subset X$ which is the preimage of an analytic Zariski open $\mathcal{V} \subset \mathcal{Y}$, e.g., the locus where f is smooth. By induction, $W^{\text{an}} := X^{\text{an}} \setminus U^{\text{an}} \rightarrow Z^{\text{an}} := \mathcal{Y} \setminus \mathcal{V}$ is algebraic, hence the equivalence relation $W \times_Z W \subset W \times W$ is algebraic. The equivalence relation $U^{\text{an}} \times_{\mathcal{V}} U^{\text{an}} \subset U^{\text{an}} \times U^{\text{an}}$ is also algebraic, since it corresponds to a Zariski open subset of a component of the Hilbert scheme of a compactification of X . Thus, the full equivalence relation $X^{\text{an}} \times_{\mathcal{Y}} X^{\text{an}} \subset X^{\text{an}} \times X^{\text{an}}$ is algebraic. Since quotients by proper equivalence relations exist in the category of definable topological spaces, this means there

is a morphism of definable topological spaces $q: |X^{\text{def}}| \rightarrow \mathfrak{Y}$ whose underlying map of topological spaces agrees with f .

Step 3. $\mathfrak{Y}$ has a finite cover $\mathfrak{V}_i$ by contractible definable open sets. Thus, $q^{-1}(\mathfrak{V}_i)$ lifts to $\widehat{X^{\text{an}}}^{V_Z}$, and by Key Fact 3.13, $q^{-1}(\mathfrak{V}_i) \rightarrow \check{\mathbb{D}}(V_{\mathbb{C},x})$ is definable analytic with compact fibers. Hence, by definable Stein factorization, we may take the Stein factorization, which equips $\mathfrak{V}_i$ with a definable analytic space structure uniquely determined by the property that $q^{-1}(\mathfrak{V}_i) \rightarrow \mathfrak{V}_i$ is a morphism of definable analytic spaces. Thus, we obtain the desired morphism $X^{\text{def}} \rightarrow \mathcal{Y}$ in the definable analytic category.

Step 4. By definable Artin, we obtain the desired morphism $X \rightarrow Y$ in the algebraic category. $\square$

Remark 3.15. It is proven in [6] that in fact the closure of the image of φ in Theorem 3.14 is algebraic, resolving a conjecture of Griffiths. This means we may additionally take ν to be a closed immersion. This requires showing that $\mathbf{G}(\mathbb{Z}) \backslash \mathbb{D}$ has the structure of a definable analytic space for which φ is a morphism [9], and then definable Artin can be applied directly.

3.4. Quasiprojectivity of period images. It is moreover shown in [6] that the “period image” Z (that is, the closure of the image of φ in Theorem 3.14) is quasiprojective, and that the Griffiths bundle $L := \bigotimes_p \det(F^p V_{\mathcal{O}})$ on Z is ample. This hinges on the following ampleness criterion, which abstracts away the algebro-geometric properties the Griffiths bundles inherits from the negative curvature of the Hodge metric.

Theorem 3.16 ([6]). *Let X be an algebraic space and L_X a line bundle on X with the following property:*

For every reduced closed subspace $i: Z \rightarrow X$, any proper log smooth algebraic space (W, D) , and any proper birational morphism $\pi: W \setminus D \rightarrow Z$, setting $g_{W \setminus D} := i \circ \pi$, the pullback $g_{W \setminus D}^* L_X$ extends to a big and nef line bundle L_W on W . Moreover, this extension is functorial in the sense that for two such Z, Z' and $(W, D), (W', D')$, and morphism $h: (W, D) \rightarrow (W', D')$ for which $g_{W \setminus D} = g_{W' \setminus D'} \circ h$, we have $L_W \cong h^* L_{W'}$ as extensions of $g_{W \setminus D}^* L_X$.

Then L_X is ample. Moreover, the space of sections of L_X^k which extend to L_Y^k for a log smooth compactification (Y, D) of a resolution $\pi: Y \setminus D \rightarrow X$ is finite-dimensional and gives a projective embedding of X for k sufficiently large.

Remark 3.17. Theorem 3.16 should be interpreted as a Nakai–Moishezon condition for open varieties, but very special line bundles (admitting this singular extension property). Its proof relies on Fujita’s vanishing (see [6, Thm 5.4]). Ampleness criteria are more subtle in the non-proper setting—see for example [18, Proposition 2], wherein a normal crossings surface T_2 is constructed for which ω_{T_2} is not ample, but its pullback to the normalization $\pi: \bar{T}_2 \rightarrow T_2$ is. The example is obtained from a surface S by gluing two curves D_p, D_q along an isomorphism $\sigma: D_p \rightarrow D_q$ to get $T = S/\sigma$ and removing some singular points Z lying on the image D of D_p (which is also the image of D_q), so $T_2 = T \setminus Z$. The curve D lifts in two ways to S , but the two ample extensions $\pi^* \omega_T|_{D_p}$ and $\pi^* \omega_T|_{D_q}$ of $\pi^* \omega_{T_2}|_{D \setminus Z}$ are not compatible with the gluing. Thus, the functoriality required in the setup of Theorem 3.16 is not satisfied.

4. VHSS: DEGENERATIONS, TRANSCENDENTAL LATTICES, AND POLARIZATIONS

4.1. Degenerations. Degenerations of families of algebraic varieties are very complicated. On the other hand, the associated linear object, the variation of Hodge structures, has

understandable degeneration behavior which still retains a lot of interesting geometric information.

Throughout, let X be a smooth algebraic variety, $X \subset \overline{X}$ a smooth compactification with snc boundary $D = \overline{X} \setminus X$. The set of irreducible components of D is naturally identified with $\pi_0(D^{\text{reg}})$. For any $\Sigma \subset \pi_0(X^{\text{reg}})$, define

$$X_\Sigma := \bigcap_{E \in \Sigma} E \setminus \bigcup_{E \notin \Sigma} E.$$

Then $\overline{X}$ is naturally stratified by the connected components of the X_Σ , each of which has codimension $|\Sigma|$. After blowing up the closed strata $\overline{X}_\Sigma$ judiciously, each X_Σ will be connected, and this will be sufficient for our purposes.

We assume we have a polarizable $\mathbb{Z}$ -VHS V on X with unipotent local monodromy. Recall that in the previous section, the basic objects that show up are local systems $V_{\mathbb{C}}$, flat vector bundles $V_{\mathcal{O}^{\text{an}}}$, and filtered logarithmic extensions $\mathcal{V}$, and VHSs V .

4.1.1. Local systems. Each X_Σ admits a tubular neighborhood $\mathfrak{T}(X_\Sigma)$. This neighborhood has the topology of a $\Delta^{|\Sigma|}$ -bundle over X_Σ , and the punctured neighborhood $\mathfrak{T}^*(X_\Sigma) := \mathfrak{T}(X_\Sigma) \setminus D$ looks like a $(\Delta^*)^{|\Sigma|}$ -bundle over X_Σ . In particular we have an exact sequence

$$\mathbb{Z}^{|\Sigma|} \simeq \pi_1((\Delta^*)^{|\Sigma|}) \rightarrow \pi_1(\mathfrak{T}^*(X_\Sigma)) \rightarrow \pi_1(X_\Sigma) \rightarrow 1$$

where the first group is central in the second (essentially because on a Zariski open the sequence is split by defining equations for the divisors). The identification with $\mathbb{Z}^{|\Sigma|}$ is given by the small loops around each divisor E in Σ .

As a result, the local monodromies T_E give endomorphisms of $V_{\mathbb{Q}}|_{\mathfrak{T}^*(X_\Sigma)}$, as do the nilpotent logarithms N_E . We define the covariants ⁴

$$V(\Sigma)_{\mathbb{Q}}|_{\mathfrak{T}^*(X_\Sigma)} := \text{coker} \left(\bigoplus_{E \in \Sigma} N_E : \bigoplus_{E \in \Sigma} V_{\mathbb{Q}}|_{\mathfrak{T}^*(X_\Sigma)} \rightarrow V_{\mathbb{Q}}|_{\mathfrak{T}^*(X_\Sigma)} \right).$$

which is the maximal quotient on which the local monodromies act trivially. In particular, this local system extends to $V(\Sigma)_{\mathbb{Q}}$ on $\mathfrak{T}(X_\Sigma)$, which explains the notation. Note that it has a natural integral structure $V(\Sigma)_{\mathbb{Z}}$.

The Jacobson–Morozov filtration $W(\Sigma)_\bullet V_{\mathbb{Q}}|_{\mathfrak{T}^*(X_\Sigma)}$ associated to $\sum_{E \in \Sigma} N_E$ induces a filtration $W_\bullet V(\Sigma)_{\mathbb{Q}}$, and each N_E has degree -2 with respect to this filtration.

4.1.2. Log flat vector bundles. The Deligne extension is functorial, so we obtain endomorphisms N_E of the associated log flat vector bundles, and we likewise define

$$V(\Sigma)_{\mathcal{O}^{\text{an}}} := \text{coker} \left(\bigoplus_{E \in \Sigma} N_E : \bigoplus_{E \in \Sigma} \mathcal{V}|_{\mathfrak{T}(X_\Sigma)} \rightarrow \mathcal{V}|_{\mathfrak{T}(X_\Sigma)} \right).$$

which is the flat vector bundle associated to $V(\Sigma)$. It is a quotient $\mathcal{V}|_{\mathfrak{T}(X_\Sigma)} \rightarrow V(\Sigma)_{\mathcal{O}^{\text{an}}}$ in the category of log flat vector bundles, but the connection of $V(\Sigma)_{\mathcal{O}^{\text{an}}}$ has no residues. It likewise has a filtration $W_\bullet V(\Sigma)_{\mathcal{O}^{\text{an}}}$.

⁴The reason why we take coinvariants instead of invariants becomes clear in §4.3. If we project the (Hodge) line sub-bundle in $\mathcal{V}$ to a quotient of $\mathcal{V}$ and this projection is nonzero along a locally closed subset Z , then it remains nonzero in a neighborhood of Z . By contrast, if we define a subobject of $\mathcal{V}$ that contains the Hodge line when restricted to Z , there is no guarantee that it continues to contain the Hodge line in a neighborhood of Z .

4.1.3. *VHSs.* The filtration $F^\bullet \mathcal{V}$ potentially interacts weirdly with $W(\Sigma)_\bullet$, but in the limit it induces the limit mixed Hodge structure. In particular,

$$V_\Sigma := (V(\Sigma)_\mathbb{Z}|_{X_\Sigma}, W_\bullet V(\Sigma)_\mathbb{Q}|_{X_\Sigma}, F^\bullet, F'^\bullet)$$

is an admissible graded-polarizable mixed $\mathbb{Z}$ -VHS, where $F^\bullet, F'^\bullet$ are the filtrations induced by those on $\mathcal{V}$. It is the mixed variation associated to the mixed Hodge module $H^0(j_\Sigma^* j_* V^H[\dim X])$ where $j = j_\emptyset : X \rightarrow \overline{X}$ and $j_\Sigma : X_\Sigma \rightarrow \overline{X}$ are the inclusions. Importantly, there is canonically only a cone of polarizations, coming from the local monodromies and the polarization on the open stratum.

There is an important compatibility in these constructions: we may obtain V_Σ by degenerating from the open stratum as above, or by degenerating along any sequence of strata terminating in X_Σ .

4.2. CY variations and transcendental lattices. Lines subbundles of VHSs are special, at least because of Corollary 3.12. The following will play a special role in our construction:

Definition 4.1. A mixed $\mathbb{C}$ -VHS V is CY if the deepest non-zero part of the Hodge filtration is rank one. In this case, we refer to $F^{p_{\max}} V_\emptyset$ (and/or the Schmid extension $F^{p_{\max}} \mathcal{V}$) as the Hodge bundle L of V .

We have the following important positivity property, coming from the moderate growth of the hodge norm:

Lemma 4.2 ([6, Lemma 6.17]). *Let V be a CY polarizable pure $\mathbb{C}$ -VHS on a smooth algebraic space X with unipotent local monodromy and let $\overline{X}$ be a smooth compactification with snc boundary. Then the Schmid extension of the Hodge bundle is nef.*

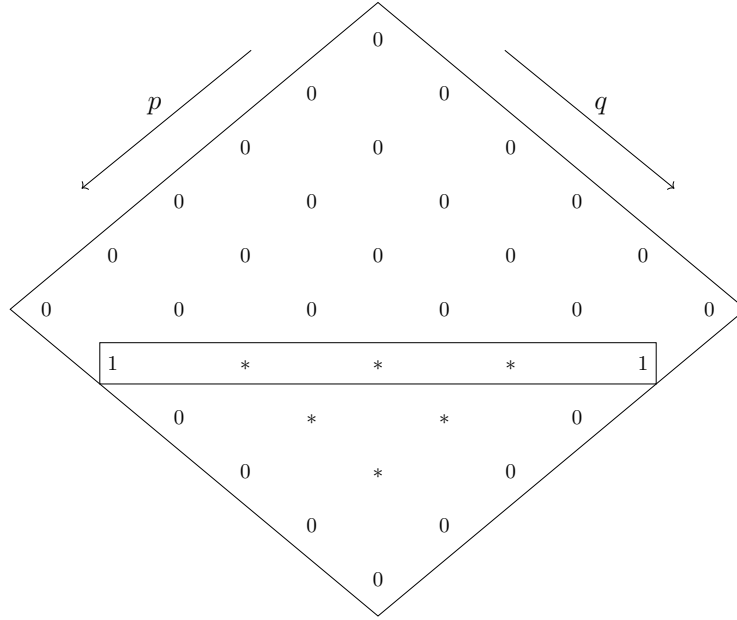
For a K3 surface S , the transcendental lattice often shows up; it is the smallest rational sub Hodge structure of $H^2(S, \mathbb{Q})$ which contains $H^{2,0}(S)$. There is a mixed version:

Definition 4.3. Let V be a CY graded-polarizable mixed $\mathbb{Q}$ -VHS. There is a minimal quotient (resp. sub) $\mathbb{Q}$ -VHS $V \rightarrow V^{\min-q}$ (resp. $V^{\min-s} \subset V$) to which the Hodge line L injects (resp. through which the Hodge line factors). It is called the CY minimal quotient (resp. sub).

Note that the Hodge line in $V^{\min-q}$ necessarily has minimal weight in $V^{\min-q}$. Moreover, we define the transcendental part

$$V^{\text{tr}} := (V^{\min-q})^{\min-s} = (V^{\min-s})^{\min-q} = W_{k_{\min}} V^{\min-q}.$$

The Hodge diamond of $V^{\min-q}$ might look like this:

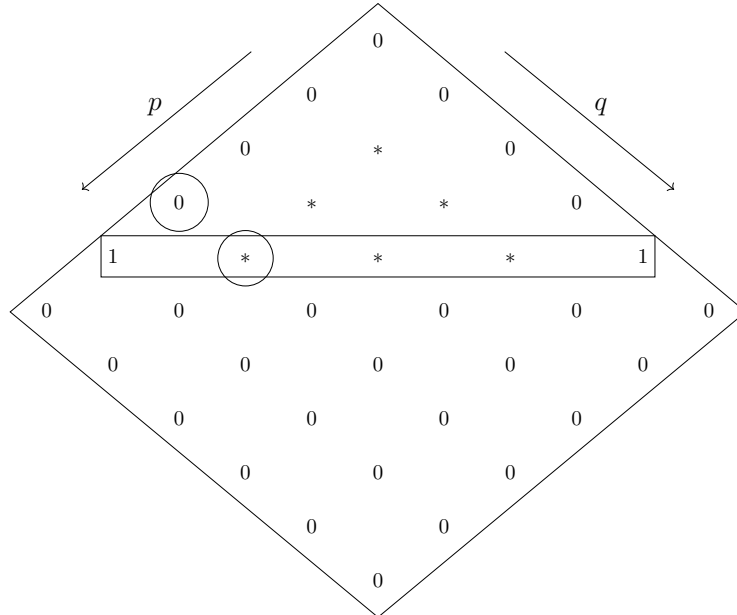


We will only use the CY minimal quotient, so we will just call it $V^{\min}$ when there is no ambiguity.

Remark 4.4. Note that as is already the case with the transcendental lattice of a K3 surface, formation of the CY minimal quotient and transcendental part does *not* commute with restriction to a subvariety, although it does for restriction to a subvariety through a very general point.

Lemma 4.5. Suppose V is a CY graded-polarizable mixed $\mathbb{Q}$ -VHS on X and that $f: X \rightarrow Y$ is a smooth morphism with connected fibers that is a topological fibration such that V^{tr} descends to Y . Then $V^{\min}$ also descends to Y .

Proof. The proof is best phrased in terms of $(V^{\min})^\vee$ and described in pictures. The Hodge diamond of $(V^{\min})^\vee$ might look like:



The target of the connection applied to the Hodge line lands in the two circled groups. It follows that if Z is a fiber of f , then since the Hodge line of $(V^{\text{tr}})^{\vee}|_Z$ is flat with trivial monodromy, so is the Hodge line of $(V^{\text{min}})^{\vee}|_Z$. By the theorem of the fixed part, the fixed part of $(V^{\text{min}})^{\vee}|_Z$ is a sub-variation containing the Hodge line, hence $((V^{\text{min}})^{\vee}|_Z)^{\text{min-s}}$ has trivial monodromy. Taking Z to be a very general fiber, it follows that $V^{\text{min}}|_Z$ has trivial monodromy. $\square$

4.3. Quasiperiod maps. The main result (see Theorem 5.1) will be a semiampleness criterion for the Schmid extension of the Hodge bundle in a CY polarizable pure $\mathbb{Z}$ -VHS with unipotent local monodromy. After some preparation, the variations V_{Σ}^{tr} on each stratum X_{Σ} will give a Stein-factorized period map $X_{\Sigma} \rightarrow Y_{\Sigma}$ as in Theorem 3.14 for which the Hodge bundle descends and is ample (provided an integrability condition is satisfied). The key step—and the most difficult part of the proof of Baily–Borel in the classical case [5]—is to glue these algebraic structures, which requires lifting the algebraic coordinates, or sections of the Hodge bundle. However, whereas the local system underlying V_{Σ}^{tr} lifts to a tubular neighborhood of X_{Σ} in X as a subquotient of the local system $V_{\mathbb{Q}}$, the Hodge bundle of V will *not* in general land in this subquotient off of X_{Σ} . By contrast, the Hodge bundle *will* give a line subbundle of the flat extension of the quotient V_{Σ}^{min} , which is naturally a quotient of $\mathcal{V}$.

Lemma 4.6. *Let $x_{\Sigma} \in X_{\Sigma}$ be a basepoint. By flatly transporting the Hodge bundle, we have a commutative diagram whose morphisms are definable analytic in the sense of Key Fact 3.13*

$$\begin{array}{ccccccc}
 \widetilde{\mathfrak{T}}(X_{\Sigma})^{V^{\text{min}}(\Sigma)_{\mathbb{Q}}} & \xrightarrow{\hspace{2cm}} & & & \mathbb{P}(V_{\Sigma, \mathbb{C}, x_{\Sigma}}^{\text{min}})^{\text{an}} \\
 \uparrow & & & & \uparrow \\
 \widetilde{X}_{\Sigma}^{V_{\Sigma, \mathbb{Q}}^{\text{min}}} & \xrightarrow{\hspace{1cm}} & \widetilde{X}_{\Sigma}^{V_{\Sigma, \mathbb{Q}}^{\text{tr}}} & \xrightarrow{\hspace{1cm}} & \check{\mathbb{D}}(V_{\Sigma, \mathbb{C}, x_{\Sigma}}^{\text{tr}}) & \xrightarrow{\hspace{1cm}} & \mathbb{P}(V_{\Sigma, \mathbb{C}, x_{\Sigma}}^{\text{tr}})^{\text{an}} \\
 & \searrow & & \searrow & & \nearrow & \\
 & & \widetilde{Y}_{\Sigma}^{V_{\Sigma, \mathbb{Q}}^{\text{min}}} & \xrightarrow{\hspace{1cm}} & \widetilde{Y}_{\Sigma}^{V_{\Sigma, \mathbb{Q}}^{\text{tr}}} & &
 \end{array}$$

Crucially, Lemma 4.5 tells us that while $V_{\Sigma, \mathbb{Q}}^{\text{min}}$ might have more monodromy than $V_{\Sigma, \mathbb{Q}}^{\text{tr}}$, the two negative slope diagonal maps above have the same fibers. If the connected components of the fibers were compact, we could try to apply definable Stein factorization as in Section 3.3. In general, we must glue together different strata X_{Σ} before the “quasiperiod maps” $\widetilde{X}_{\Sigma}^{V_{\Sigma, \mathbb{Q}}^{\text{tr}}} \rightarrow \mathbb{P}(V_{\Sigma, \mathbb{C}, x_{\Sigma}}^{\text{tr}})^{\text{an}}$ has fibers with compact connected components.

We have the following positivity property complementing Lemma 4.2, coming from the curvature of the hodge metric:

Lemma 4.7 ([6, Lemma 6.17]). *Let V be a CY polarizable pure $\mathbb{C}$ -VHS on a smooth algebraic space X with unipotent local monodromy and let $\overline{X}$ be a smooth compactification with snc boundary. Then the Schmid extension of the Hodge bundle is big iff the quasiperiod map $\widetilde{X}^{V_{\mathbb{C}}} \rightarrow \mathbb{P}(V_{\mathbb{C}, x})$ is generically immersive.*

In particular, since if the quasiperiod map is nontrivial we can always find a curve on which it is nontrivial, we deduce:

Corollary 4.8. *Let V be a CY polarizable pure $\mathbb{C}$ -VHS on a smooth algebraic space X with unipotent local monodromy and let $\bar{X}$ be a smooth compactification with snc boundary. Then the Schmid extension of the Hodge bundle is numerically trivial iff it is flat.*

4.4. Polarizations. One subtle but important aspect of the degeneration package described in Section 4.1 is that the variations V_Σ on each stratum are graded-polarizable, but the polarization depends on the local monodromy. In particular, when we glue together different strata to get fibers of $\bar{X} \rightarrow Y^{\text{BB}}$, the boundary variations will glue but without a global polarization. This will be important since we must first show the map $\bar{X} \rightarrow Y^{\text{BB}}$ is well-behaved on the level of topological spaces—that is, that the equivalence relation defining the map is proper. While we essentially know this on each stratum since the map is given by a period map, pathologies could arise a priori if the gluings force unbounded identifications on a stratum by stringing together equivalences with points in other strata.

We therefore must understand the difference between isomorphism classes of polarizable $\mathbb{Z}$ -Hodge structures, and those of *polarized* $\mathbb{Z}$ -Hodge structures. Theorem 4.9 below has the same proof as a theorem of Narasimhan–Nori.

Let $\mathbb{D}$ be a Mumford–Tate domain, which is essentially the space of polarizable Hodge structures on a fixed lattice $V_{\mathbb{Z}}$ for which fixed tensors are Hodge classes. Up to connected components, $\mathbb{D}$ is an orbit for $\mathbf{G}(\mathbb{R})$ where $\mathbf{G} \subset \mathbf{GL}(V_{\mathbb{Q}})$ is the stabilizer of those tensors. The space $\mathbf{G}(\mathbb{Z}) \backslash \mathbb{D}$ parametrizes isomorphism classes of such $\mathbb{Z}$ -Hodge structures via isomorphisms that fix those tensors. On the other hand, for $V \in \mathbf{G}(\mathbb{Z}) \backslash \mathbb{D}$ we may consider

$$O_V := \{V' \in \mathbf{G}(\mathbb{Z}) \backslash \mathbb{D} \mid V \cong V' \text{ as } \mathbb{Z}\text{-Hodge structures}\}.$$

Theorem 4.9 (Narasimhan–Nori [23] + ϵ [8]). *For every $V \in \mathbf{G}(\mathbb{Z}) \backslash \mathbb{D}$, O_V is finite.*

The statement is a bit subtle—for instance, the size of O_V is unbounded.

Sketch of proof. One easily reduces to the case that $\mathbb{D}$ is a full polarized period domain (for which the only required Hodge tensor is a fixed polarization). Let $P_{\mathbb{Z}}$ (resp. $P_{\mathbb{Q}}$) be the set of polarizations of V as a $\mathbb{Z}$ - (resp. $\mathbb{Q}$ -) Hodge structure, and fix $q_0 \in P_{\mathbb{Z}}$. For any $q \in P_{\mathbb{Q}}$, denote $\psi_q: V_{\mathbb{Q}} \rightarrow V_{\mathbb{Q}}^{\vee}$ the induced isomorphism. Let $B_{\mathbb{Z}} = \text{End}_{\mathbb{Z}\text{-Hodge}}(V)$ be the algebra of endomorphisms of V as a $\mathbb{Z}$ -Hodge structure. Note that $B_{\mathbb{Q}}$ has a natural involution $f \mapsto f^{\dagger} = \psi_{q_0}^{-1} f^{\vee} \psi_{q_0}$ where $f^{\dagger}$ is the adjoint with respect to q_0 , or equivalently the hodge form induced by q_0 (over $\mathbb{C}$). It thus follows that $B_{\mathbb{Q}}$ is semisimple; let $\mathbf{H}$ be the $\mathbb{Q}$ -algebraic group of units of $B_{\mathbb{Q}}$, and note $(-)^{\dagger}$ gives an involution $\mathbf{H} \rightarrow \mathbf{H}^{\text{op}}$.

Let $A_{\mathbb{Q}} \subset B_{\mathbb{Q}}^{\text{sa}}$ be the subalgebra of self-adjoint elements. There is a natural injection $\iota: P_{\mathbb{Q}} \rightarrow B_{\mathbb{Q}}$ given by $q \mapsto \psi_q^{-1} \psi_{q_0}$ which factors through $A_{\mathbb{Q}}$ and is equivariant with respect to the natural left action by $\mathbf{H}(\mathbb{Q})$ on the source by $g: q \mapsto (g^{-1})^* q$ and the action $g: f \mapsto (g^{-1})^{\dagger} f g^{-1}$ on the target. If $\mathbf{H}(\mathbb{Z}) = \mathbf{H}(\mathbb{Q}) \cap B_{\mathbb{Z}}$ is the lattice of $\mathbb{Z}$ -Hodge automorphisms, and $A_{\mathbb{Z}}$ the minimal lattice of $A_{\mathbb{Q}}$ containing $\iota(P_{\mathbb{Z}})$, then $\mathbf{H}(\mathbb{Z})$ preserves $A_{\mathbb{Z}}$. One checks that the orbits of $\mathbf{H}$ on $\Xi := \{f \in A_{\mathbb{C}} \mid \det f = 1\}$ are finite in number and Zariski closed [23, Lem. 3.1], in which case it follows from the theorem of Borel–Harish-Chandra that $\Xi(\mathbb{Q}) \cap A_{\mathbb{Z}}$ has finitely many $\mathbf{H}(\mathbb{Z})$ -orbits. This implies V admits finitely many $\mathbf{H}(\mathbb{Z})$ -isomorphism classes of $\mathbb{Z}$ -polarizations q of the same discriminant as q_0 , which implies the claim. $\square$

5. THE SEMIAMPLeness CRITERION

The main result of [8] is a semiamplessness criterion for line bundle coming from Hodge theory.

Theorem 5.1. [8, Thm 4.1] *Let X be a smooth algebraic variety, $X \subset \overline{X}$ a smooth compactification with snc boundary $D = \overline{X} \setminus X$. Let V be a polarizable pure $\mathbb{Z}$ -VHS on X with unipotent local monodromy.*

Then the nef line bundle $L := \bigotimes_p (\det F^p V)^{a_p}$ on $\overline{X}$, with $a_p \geq 0$, is semiample if and only if it is integrable with torsion combinatorial monodromy.

We first discuss the two conditions in Theorem 5.1, exhibit a nef line bundle which does not enjoy the two conditions in §5.3, and finally provide the strategy of the proof in §6.

5.1. Combinatorial monodromy. A semiample line bundle N induces a morphism $q: \overline{X} \rightarrow (\overline{Z}, \mathcal{O}_{\overline{Z}}(1))$ with $q^* \mathcal{O}_{\overline{Z}}(1) \simeq N^k$. Therefore, the following property holds:

- (†) if $N|_W \equiv 0$ on a reduced closed algebraic subspace $W \subset \overline{X}$, then $N|_W$ is a torsion line bundle,

because any connected component of W must be contained in a fiber of q .

The condition (†) always holds true for $N = L$ provided that W is irreducible. Indeed, in that case, the generic point of W is contained in some X_Σ , and by Corollary 4.8, $L|_W$ is a flat subbundle of the Deligne extension of the flat bundle underlying the polarizable $\mathbb{Z}$ -VHS $V_\Sigma^{\text{tr}}|_{W \cap X_\Sigma}$, which must be torsion by Deligne's theorem Corollary 3.12.

If W is reducible, the condition (†) is not automatic and must be checked. Since it holds for each irreducible component, this amounts to a subtle condition on their combinatorics. If $\overline{X}$ is an algebraic space, one can reduce to test the condition along curve, or rather snc curve mapping to $\overline{X}$; see [8, Lem. 3.14].

Definition 5.2. A line bundle L on $\overline{X}$ has **torsion combinatorial monodromy** if $g^* L$ is torsion along any connected (snc) curve $g: C = \bigcup_j C_j \rightarrow \overline{X}$ with $\deg(g^* L|_{C_j}) = 0$.

5.2. Integrability. The integrability of the line bundle L can be understood informally as the condition that L varies if the associated VHS $\bigotimes_p (\wedge^{\text{rank } F^p V} V)^{\otimes a_p}$ varies.

Definition 5.3. The line bundle L is **integrable** if one of the following equivalent condition holds:

- (1) if $L|_T$ is flat, as a subbundle of $(\overline{V}_\mathcal{O}, \nabla)$, on a reduced analytic germ $T \subset \overline{X}$, then $L|_{\overline{T}^{\text{Zar}}}$ is flat;
- (2) the leaves of the foliation associated to the quasiperiod maps in §4.3 are algebraic;
- (3) if the period map of $((V_\Sigma^{\text{tr}})|_{W \cap X_\Sigma})^{\text{tr}}$ on (a log resolution of) a reduced irreducible closed algebraic subspace $W \subset \overline{X}$, whose generic point is contained in X_Σ , is generically immersive, then $L|_W$ is big.

Proof of the equivalence. (1) $\implies$ (2): clear;

(2) $\implies$ (3): If $L|_W$ is not big and the leaves of the foliation are algebraic, then through a very general point of W there is an algebraic curve $C \subset W$ such that $L|_C$ is flat and torsion, so the quasiperiod map is constant along C , so not generically immersive on W ;

(3) $\implies$ (1): If $L|_{\overline{T}^{\text{Zar}}}$ is not flat, up to taking complete intersections on a projective model of $\overline{X}$, we can suppose that the very general leaf of the foliation is 1-dimensional and non-algebraic. So $L|_{\overline{T}^{\text{Zar}}}$ is strictly nef, not big. $\square$

If L is semiample, then T is contained in a fiber of q , and so does its Zariski closure $\overline{T}^{\text{Zar}}$. Hence, condition (1) holds.

5.3. Non-example. We exhibit a CY polarizable pure $\mathbb{Z}$ -VHS whose Hodge bundle is neither integrable nor with torsion combinatorial monodromy. Let $F := \mathbb{Q}[\sqrt{d}]$ be a real quadratic field over $\mathbb{Q}$ with two real embeddings $\sigma_1, \sigma_2: F \hookrightarrow \mathbb{R}$ with ring of integers $\mathcal{O}_F$. The moduli space X of abelian surfaces A with real multiplication $F \hookrightarrow \text{End}_{\mathbb{Q}}(A)$ can

be identified with the quotient $\mathrm{SL}_2(\mathcal{O}_F) \backslash \mathbb{H} \times \mathbb{H}$ where $\mathrm{SL}_2(\mathcal{O}_F)$ acts on the self-product of the open upper half-plane $\mathbb{H} \subset \mathbb{C}$ (or bidisk) by ordinary Moebius action, performed simultaneously under the two embeddings

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau_1, \tau_2) = \left(\frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \frac{\sigma_2(a)\tau_2 + \sigma_2(b)}{\sigma_2(c)\tau_2 + \sigma_2(d)} \right).$$

Indeed, there is an identification assigning to the pair (τ_1, τ_2) the abelian surface

$$A = A(\tau_1, \tau_2) \simeq \mathbb{C}^2 / (\sigma_1(a) + \sigma_1(b)z_1, \sigma_2(a) + \sigma_2(b)z_1)$$

for $a, b \in \mathcal{O}_F$. As F -module, the first real cohomology group splits into the sum of real Hodge structures of elliptic curves

$$H^1(A, \mathbb{R}) = F^2 \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^2 \oplus \mathbb{R}^2 = V_1 \oplus V_2, \\ x \otimes 1 \mapsto (\sigma_1(x), \sigma_2(x))$$

where V_i are the σ_i -eigenspaces for the action of F on $H^1(A, \mathbb{R})$. Therefore, $H^1(A, \mathbb{Z})^{\oplus 2}$ admits a polarized integral Hodge structure of weight 5 with non-standard Hodge decomposition

$$H^{0,5} = V_1^{0,1} \quad H^{1,4} = V_1^{1,0} \quad H^{2,3} = (V_2^{0,1})^{\oplus 2} \quad H^{3,2} = (V_2^{1,0})^{\oplus 2} \quad H^{4,1} = V_1^{0,1} \quad H^{5,0} = V_1^{1,0}.$$

The associated polarized pure $\mathbb{Z}$ -VHS V on X is Calabi–Yau, with Hodge bundle λ given by $\lambda_x = V_{1,x}^{1,0}$ for all $x \in X$.

Lemma 5.4. *λ is not integrable.*

Proof. Let T the image along the quotient in X of a fiber of the first projection $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}$. T is a Zariski-dense analytic subspace in X . By construction, $\lambda_T \equiv 0$, but $\lambda_{\overline{T}^{\mathrm{Zar}}} = \lambda_X \neq 0$, or the period map is an isomorphism. $\square$

Let ν be the line bundle whose fiber $\nu_x = V_1^{1,0} \otimes V_2^{1,0}$, i.e., the Hodge bundle of polarized integral variation of the standard Hodge structure $H^2(A, \mathbb{Z})$. The Baily–Borel compactification $X^{\mathrm{BB}} := \mathrm{Proj} \bigoplus_i H^0(X, \nu^i)$ is a projective compactification of the surface $X = \mathrm{PSL}_2(\mathcal{O}_F) \backslash \mathbb{H} \times \mathbb{H}$, obtained by adding isolated points, called cusps. Here we have written $\mathrm{PSL}_2(\mathcal{O}_F) = \mathrm{SL}_2(\mathcal{O}_F) / \{\pm I\}$, since the center of $\mathrm{SL}_2(\mathcal{O}_F)$ acts trivially on $\mathbb{H} \times \mathbb{H}$. One of these cusps can be identified with the point $(\infty, \infty) \in \overline{\mathbb{H}} \times \overline{\mathbb{H}}$. A punctured neighbourhood of $(\infty, \infty) \in X^{\mathrm{BB}}$ is modelled on the quotient of $\mathbb{U}_{(\infty, \infty)}^* := \{(\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H} : \tau_i > N_i\}$, with $N_i \gg 0$, by the parabolic stabilizer of (∞, ∞) in $\mathrm{PSL}_2(\mathcal{O}_F)$

$$\mathrm{Stab}_{(\infty, \infty)} \mathrm{SL}_2(\mathcal{O}_F) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathcal{O}_F^*, b \in \mathcal{O}_F \right\} / \{\pm I\}.$$

The stabilizer fits in the exact sequence

$$1 \rightarrow U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \rightarrow \mathrm{Stab}_{(\infty, \infty)} \mathrm{SL}_2(\mathcal{O}_F) \rightarrow L \rightarrow 1,$$

where

$$L := \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \simeq \mathcal{O}_F^\times / \{\pm 1\} \simeq \mathbb{Z}\epsilon;$$

e.g., $\epsilon = (1 + \sqrt{5})/2$ for $d = 5$. A neighborhood of $(\infty, \infty) \in X^{\mathrm{BB}}$ is a surface cusp singularity whose minimal resolution consists of a cycle $C = \bigcup_j C_j$ of rational curve, which, as a loop, corresponds to the infinite generator of L . Indeed, since U acts by translation on $\mathbb{H}$, then $\mathbb{U}_{(\infty, \infty)}^* / U$ is modeled on a neighborhood of $(0, 0) \in \mathbb{C} \times \mathbb{C}$, intersected with the algebraic torus $\mathbb{C}^* \times \mathbb{C}^*$, endowed with the residual action of L . The cycle of rational curves comes from the L -quotient of a toric L -equivariant partial compactification of $\mathbb{U}_{(\infty, \infty)}^* / U \subset$

$\mathbb{C}^* \times \mathbb{C}^*$, i.e., an infinite chain of toric blowups of $\mathbb{C} \times \mathbb{C}$ whose exceptional locus is an infinite chain of rational curves, translated by L , called Tate curve. See [4, §1.5] for more detail. With a slight abuse of notation, let λ be the Schmid's extension of the Hodge bundle of V on a minimal resolution $\pi: \bar{X} \rightarrow X^{\text{BB}}$.

Lemma 5.5. *λ does not have torsion combinatorial monodromy along $C = \bigcup_j C_j$.*

Proof. By construction, $\pi^*\nu = \lambda \otimes \lambda'$, where λ and λ' are both semipositive. Therefore, since $\pi^*\nu = 0$ along C , we have $\lambda|_{C_j} \equiv 0$. The topological loop generating the fundamental group of C can be identified with the generator of L , acting on $\mathbb{H}$ via the Moebius transformation $\epsilon\tau_i/\epsilon^{-1} = \epsilon^2\tau_i$, and so on the fiber of the Hodge bundle at $c \in C$ by the complex number $\sigma_1(\epsilon^2)$, which has infinite order. $\square$

Remark 5.6. The product structure of $\mathbb{H} \times \mathbb{H}$ does not descend to X , but instead induces two foliations on X . The Hodge bundle λ is the canonical bundle of one of this foliation. The fact that it is not semiample is a classical example of the failure of foliated abundance; see [22, Ex. IV.5.5].

Remark 5.7. In fact, the above example can be modified to produce a CY polarizable pure $\mathbb{Z}$ -VHS whose Hodge bundle is: integrable but does not have torsion combinatorial monodromy; has torsion combinatorial monodromy but is not integrable; see [8, §4.2].

5.4. Geometric case. Let $f: Y \rightarrow X$ be a smooth family of K-trivial manifolds of dimension n between algebraic spaces. By semistable reduction and MMP, up to an alteration of the base, f admits a compactification $\bar{f}: \bar{Y} \rightarrow \bar{X}$, with $K_{\bar{Y}/\bar{X}} \sim_{\bar{f}} 0$, snc discriminant D , dlt in codimension one, and more generally with semi-log canonical fibers.

Theorem 5.8 (b-semiampleness conjecture). [8, Thm 5.2] *The Hodge bundle $\lambda := \bar{f}_*\omega_{\bar{Y}/\bar{X}}$ is semiample.*

Remark 5.9. The result holds more generally for a klt pair $(\bar{Y}, \bar{\Delta})$ with real coefficient and $K_{\bar{Y}} + \bar{\Delta} \sim_{\mathbb{R}, f} 0$.

Proof. By Theorem 5.1, it suffices to check that the Hodge bundle is integrable with torsion combinatorial monodromy.

Sources of a degeneration. We start by providing a geometric characterization V_{Σ}^{tr} in terms of the sources of $\bar{f}$.

Definition 5.10. A source $(S, \Delta_S) \rightarrow X_{\Sigma}$ of $\bar{f}$ is a minimal stratum among those dominating X_{Σ} of $Y_{X_{\Sigma}} \rightarrow X_{\Sigma}$,⁵ which is klt log Calabi–Yau by adjunction.

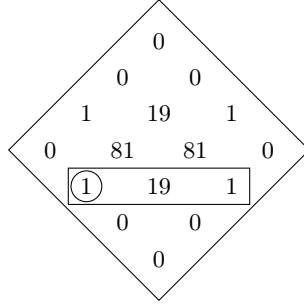
For simplicity, suppose that $\Delta_S = 0$ and $g: S \rightarrow X_{\Sigma}$ is a topologically locally trivial fibration of relative dimension m . Then V_{Σ}^{tr} is the family of the transcendental part of middle cohomology of the sources S_x with $x \in X_{\Sigma}$, i.e.,

$$V_{\Sigma}^{\text{tr}} \simeq (R^m g_* \mathbb{Z}_S)^{\text{tr}};$$

see [8, Thm 6.34].

Example 5.11. Let $\bar{f}: \bar{Y} \rightarrow (\bar{X}, D) := (\mathbb{D}, 0)$ be a degeneration over a disk of a quintic threefold to the snc union of a hyperplane H and a quartic threefold Q . The source of $\bar{f}$ over 0 is the intersection $S := H \cap Q$, i.e., a quartic K3 surface. The middle cohomology of the general fiber has Hodge numbers $(1, 101, 101, 1)$. The Hodge diamond $h^{p,q} := \dim(W_{p+q} H_{\text{lim}}^3)^{p,q}$ of the limiting mixed Hodge structure (§4.2) is

⁵More precisely, of a dlt modification of the normalization $Y_{X_{\Sigma}}$.



The circled entry, namely $h^{3,1}$, corresponds to the Hodge line. By the Clemens–Schmid sequence, the top non-vanishing row can be identified with the primitive cohomology of S , and so does the bottom one by hard Lefschetz, i.e., the logarithmic monodromy induces an isomorphism $N: \mathrm{gr}_2^W H_{\mathrm{lim}}^3 \rightarrow \mathrm{gr}_4^W H_{\mathrm{lim}}^3$. We obtain that $V_0^{\mathrm{tr}} \simeq H^2(S, \mathbb{Z})^{\mathrm{tr}}$.

Integrability. If Y_x is a smooth fiber of dimension n , then the quasiperiod map from the germ $(X, x) \rightarrow \mathbb{P}(V_x)$ factors through the space $\mathrm{Def}(Y_x)$ of local deformations of Y_x , and the map $\mathcal{P}: \mathrm{Def}(Y_x) \rightarrow \mathbb{P}(V_x)$ is generically immersive by the local Torelli theorem. Indeed, the differential

$$d\mathcal{P}_{[X_y]}: T_{[Y_x]} \mathrm{Def}(Y_x) \simeq H^1(Y_x, T_{Y_x}) \rightarrow \mathrm{Hom}(H^{n,0}(Y_x), H^{n-1,1}(Y_x)) \subseteq T_{\mathcal{P}([X_y])} \mathbb{P}(V_x)$$

can be identified with the contraction with the global volume form, which is the isomorphism $H^1(Y_x, T_{Y_x}) \simeq H^1(Y_x, \Omega_{Y_x}^{n-1})$ by the Calabi–Yau assumption. A similar argument works for klt pairs (X_y, Δ_y) ; see [1, §2]. Therefore, if $\lambda|_T$ is flat on a reduced analytic subspace $T \subset Y$, the source of $X_T \rightarrow T$ is isotrivial. Since the isotriviality of a family of klt pairs is a Zariski closed condition, and it certainly implies the flatness of the Hodge bundle, we conclude that $\lambda|_{\overline{T}^{\mathrm{Zar}}}$ is flat.

Torsion combinatorial monodromy. Fix a very general point $x \in X_\Sigma$ and let $S \rightarrow X_\Sigma$ be the source of $\bar{Y}_{X_\Sigma} \rightarrow X_\Sigma$ of relative dimension m . Mind that in general the source is a pair (S, Δ_S) , but here for simplicity we assume that there is no boundary. The fiber of the Hodge bundle at x can be identified with the global sections of the canonical bundle of S_x :

$$\lambda_x = (V_\Sigma^{\mathrm{tr}})_x^{m,0} = H^{m,0}(S_x) = H^0(S_x, \omega_{S_x}).$$

For any connected snc curve $g: C = \bigcup_j C_j \rightarrow X$ with $\deg(g^*L|_{C_j}) = 0$, we show that the monodromy on $\lambda_x = H^0(S_x, \omega_{S_x})$, for $x \in C_j \subset X_\Sigma$, along C is finite. Indeed, over the irreducible components of C , the source S_{C_j} is isotrivial; cf. proof of integrability. Over a node of C , two different sources are connected by strata birational to " $\mathbb{P}^1$ -bundle" over the source itself; see the theory of $\mathbb{P}^1$ -linking by Kollár [20, Prop. 4.37]. Overall, this implies that a monodromy operator on $H^0(S_x, \omega_{S_x})$ is induced by a birational automorphism $S_x \dashrightarrow S_x$, preserving the volume form. A resolution of indeterminacy $S_x \leftarrow \tilde{S}_x \rightarrow S_x$ induces morphisms of Hodge structures $H^m(S_x, \mathbb{Z}) \rightarrow H^m(\tilde{S}_x, \mathbb{Z}) \leftarrow H^m(S_x, \mathbb{Z})$, compatible with cup product, and restricting to an isomorphism on $H^{m,0}(S_x)$. Hence, the birational automorphism $S_x \dashrightarrow S_x$ induces a polarized automorphism of the polarized integral Hodge structure $H^m(S_x, \mathbb{Z})^{\mathrm{tr}}$. Here, the polarization of $H^m(S_x, \mathbb{Z})^{\mathrm{tr}}$ is given by the cup product. Since the automorphism preserves the polarization, its action on $H^{m,0}(S_x)$ has norm one. As it is defined over $\mathbb{Z}$, this action is given by an algebraic integer. It follows that the automorphism has finite order by Kronecker's theorem; see finiteness of b-representations in [17, Thm 1.2] or [20, §10.5]. $\square$

Theorem 5.12. [8, Thm 5.2] *In the notation of Theorem 5.1, the Griffiths bundle $L := \bigotimes_p(\det F^p\mathcal{V})$, i.e., $a_p = 1$, on $\bar{X}$ is semiample.*

Proof. By Theorem 5.1, it suffices to check that the Griffiths bundle is integrable with torsion combinatorial monodromy.

Integrability. The Griffiths bundle is ample on the image of period maps by [6], thus giving Definition 5.3.(3).

Torsion combinatorial monodromy. The Griffiths bundle is conjugate self-dual, i.e., the quadratic form on V induces an isomorphism $L \simeq \bar{L}^\vee$. Then for any connected snc curve $g: C = \bigcup_j C_j \rightarrow Y$ with $\deg(g^*L|_{C_j}) = 0$, the monodromy on L_x , with $x \in C_j \subset X_\Sigma$, has norm one. In particular, it preserves any polarization on $(V_\Sigma^{\text{tr}}|_{C_j \cap X_\Sigma})^{\text{tr}}$ (see [8, Claim 2.26]), so it has finite order; see [16, Thm 5.12]. $\square$

Question 5.13. Apply the semiamplessness criterion to other geometric contexts, e.g., hypersurfaces Y in $\mathbb{P}^N$ with $V_\mathbb{Z} = \Lambda^{h^0(\omega_Y)} H^{N-1}(Y, \mathbb{Z})$ and λ Hodge bundle.

6. STRATEGY OF THE PROOF OF THE SEMIAMPLENESS CRITERION

If L was semiample, it would define an algebraic morphism

$$q: \bar{X} \rightarrow \bar{Z} = \text{Proj} \bigoplus_i H^0(\bar{X}, L^i) = \bar{X}(\mathbb{C})/R$$

with $R \subset \bar{X}(\mathbb{C}) \times \bar{X}(\mathbb{C})$ is the equivalence relation defined by xRy if there is a proper connected curve $g: C = \bigoplus_j C_j \rightarrow \bar{X}$ with $x, y \in g(C)$ for which $\deg(g^*L|_{C_j}) = 0$.

Step 1. We first construct the contraction $q: \bar{X} \rightarrow \bar{Z}$ topologically. This amounts to show that R is algebraic. In fact, this implies that $q: \bar{X}(\mathbb{C}) \rightarrow \bar{X}(\mathbb{C})/R$ can be upgraded to a morphism of definable topological spaces as in the proof of Theorem 3.14.

Step 2. Algebraize q on larger and larger open sets of $\bar{X}$. Up to blowing up $\bar{X}$, there is an exhaustion of open sets

$$X^0 = \bar{X} \setminus D \subseteq X^{\leq 1} \subseteq \dots \subseteq X^{\leq r}$$

with X^i union of open strata of $(\bar{X}, D)$, saturated with respect to q , proper over $Z^i = q(X^i) \subset \bar{Z}$, and $X^{\leq i} = \bigcup_{j \leq i} X^j$.

Induction on the stratification index i . Suppose that the following property holds for i : q is algebraic on X^i and L separates q -fibers, i.e., global sections of L^k , vanishing along $X^{>i}$, embeds $Z^{\leq i}$ in $\mathbb{P}(H_{\text{van}}^0(X^{\leq i}, L^k))$. If one show that q is definable analytic in $\mathbb{R}_{\text{an}, \text{exp}}$ on $X^{\leq i+1}$, then by o-minimal GAGA and [6], the property above works for $i+1$ too. The base case for $i=0$ follows from [6, Thm 5.4].

Remark 6.1 (The role of o-minimality). The proof is paradigmatically different from the standard approaches to semiamplessness, e.g., Kodaira embedding theorem, the base point free theorem, the Keel semiamplessness theorem, or the Angehrn–Siu theorem. In these approaches, one typically uses cohomological methods to lift global sections from proper algebraic subspace which can potentially support base locus. The properness plays a crucial role in these arguments: primally, it ensures the finiteness properties of cohomology groups that underlie the relevant vanishing and extension arguments. In our setting, by contrast, we algebraize the morphism induced by the putative semiample line bundle on increasingly large open subsets, exploiting the flexibility of o-minimal GAGA, that does not require properness.

Remark 6.2 (Integrability and torsion combinatorial monodromy). We comment on how the assumptions of integrability and torsion combinatorial monodromy of L enter in the

proof. First, the proof of the algebraicity of R uses the integrability of L . Second, we construct a definable analytic morphism whose Stein factorization locally models q . In order to take its Stein factorization, we show the properness of the morphism: the compactness of the irreducible components of the fibers depends on the integrability; the torsion combinatorial monodromy rules out that the fibers have infinitely many connected components.

The torsion combinatorial monodromy grants also that a power of L descends to $Z^{\leq i}$, the integrability (and the functoriality of the Schmid's extension) grants that for any log resolution $\overline{W}$ of a compactification of a reduced algebraic subspace $W \subset Z^{\leq i}$, $L|_W$ extends to a nef and big line bundle on $\overline{W}$. By Nakai–Moishezon for open varieties (Theorem 3.16), this entails that L on $Z^{\leq i}$ is ample, or more precisely that sections of L^k on $\overline{X}$, vanishing along $X^{>i}$, embeds $Z^{\leq i}$ in $\mathbb{P}(H_{\text{van}}^0(X^{\leq i}, L^k))$.

Remark 6.3 (Glueing period maps). In contrast with an arbitrary semiample line bundle, the Hodge theoretic nature of L provides a guess of what the morphism q is. Naively, q should be obtained by glueing together the period maps of the polarizable $\mathbb{Z}$ -VHS V_{Σ}^{tr} , which are algebraic by [6]. This strategy is naive for at least two reasons:

- (1) Glueing the fibers of algebraic morphism defined on a stratification of $\overline{X}$ may lead to equivalence classes of infinite type. For instance, glue two copies of $\mathbb{P}^1 \times E$, with E elliptic curve, along the sections $\{0\} \times E$ via the identity and along the sections $\{\infty\} \times E$ via a translation by a non-torsion element of E . The cycle obtained by following the ruling of the two copies of $\mathbb{P}^1 \times E$ is an infinite chain of rational curves.
- (2) It is not sufficient to prescribe a morphism by its restriction on locally closed subsets, but we must prescribe it on neighbourhood of them and check glueing along overlaps. This would require extending the period maps of V_{Σ}^{tr} to a neighbourhood of X_{Σ} . However, away from X_{Σ} , the weight and Hodge filtrations do not interact in a way that defines a mixed Hodge structure, and V_{Σ}^{tr} does not extend to a VHS in a neighbourhood of X_{Σ} .

In the following sections, we explain how to correct the naive guess and address these potential problems.

6.1. Algebraicity of R . Two points x and y in $\overline{X}$ are R -equivalent if they are connected by a chain of curves $C = \bigcup_j C_j$ along which L has degree zero. The generic point of C_j is contained in a stratum of $(\overline{X}, D)$ denoted X_{Σ_j} . Recall that X_{Σ} carries the polarizable $\mathbb{Z}$ -VHS V_{Σ}^{tr} , and that its period map is algebraized by a morphism $X_{\Sigma} \rightarrow Y_{\Sigma}$. The R -orbit on each irreducible stratum X_{Σ} coincides with the (algebraic) fibers of the period map associated to V_{Σ}^{tr} by Lemma 6.4. A priori, a global R -orbit on $\overline{X}$ may pass through X_{Σ} infinitely many times, and consists of an infinite chain of such fibers. However, its intersection with a fixed X_{Σ} would be a collection of fibers of isomorphic unpolarized Hodge structure in the same period domain by Lemma 6.5. By Theorem 4.9, there are only finitely many of them, so R -orbit are forced to be algebraic. We prove the lemmas mentioned above.

Lemma 6.4. *An R -orbit on X_{Σ} is an (algebraic) fiber of the period map associated to V_{Σ}^{tr} .*

Proof. Up to shrinking X_{Σ} , i.e., up to a log resolution of $(\overline{X}, D)$, and provided that V has neat monodromy, we can suppose that V_{Σ}^{tr} and L descends on Y_{Σ} . By the integrability, since the period map is immersive on Y_{Σ} , L is big, and so, up to shrinking X_{Σ} again, we can suppose that Y_{Σ} does not contain curves along which L is flat. Equivalently, C_j is contained in the fibers of period map. $\square$

For any $x \in X_{\Sigma}$, denote by $V^{\text{tr}}(x)$ the pointwise transcendental part $(V_{\Sigma}^{\text{tr}}|_x)^{\text{tr}}$.

Lemma 6.5. *An R -orbit on $\overline{X}$ connects points with isomorphic pointwise transcendental part as unpolarized Hodge structure.*

Proof. By Deligne, the flatness of L along C_j forces the isotriviality of the transcendental part of the $V_{\Sigma_j}^{\text{tr}}|_{X_{\Sigma_j} \cap C_j}$. It follows that $V^{\text{tr}}(c)$, with $c \in C$, forms an isotrivial not-necessarily-polarizable variation of integral pure Hodge structures $V^{\text{tr}}(C)$ which is polarizable in restriction to each irreducible component. Indeed, the polarization of $V^{\text{tr}}(C_j)$ depends on the choice of a operator in the cone generated by the logarithmic monodromy around X_{i_k} with $\Sigma = \{i_1, \dots, i_r\}$; see §4.1.3. If two adjacent curves C_j lies in different strata, there is not a uniform choice in general. This occurs already for the Hilbert modular surface (see Example 5.3): if such a uniform choice would exist along the exceptional cycle C , then $V^{\text{tr}}(C)$ would admit a polarization, and the monodromy of the Hodge bundle along C would have had norm 1. $\square$

Remark 6.6. Deligne's and Narasimhan and Nori's theorems (Cor. 3.12 and Thm. 4.9) are the fundamental finiteness results in Hodge theory that tame the equivalence relation R and ensure its algebricity.

Remark 6.7 (Quotients by proper algebraic relations). A quotient of an algebraic space by a proper algebraic relation is a definable topological space, but in general it is not an algebraic space, or equivalently, by the definable Artin theorem (Theorem 2.25), a definable analytic space.

- (1) There are a non-algebraic proper analytic maps $q: X \rightarrow Y$ for which $X \times_Y X$ is algebraic (as a scheme), but Y is not; see [6, Ex. 4.13].
- (2) There are examples of finite relations whose quotient are non-noetherian; see [19, Ex. 10].
- (3) The quotient by an algebraic relation whose source and target maps are flat and locally of finite presentation is an algebraic space (and is separated if the relation is proper); see [28, Tag 04S6]. In general, it need not be a scheme; see Hironaka's example [19, Ex. 15].
- (4) For any proper algebraic relation $R \rightrightarrows X$, there are an alteration $X' \rightarrow X$ and a proper algebraic relation $R' \rightrightarrows X'$ whose general fiber is a connected component of a fiber of $R \rightrightarrows X$, and whose quotient X'/R' is an algebraic space. Indeed, after passing to the Stein factorization of $R \rightrightarrows X$, we may assume that the general fibers are connected; a suitable component of the Hilbert scheme of X' then parametrizes these fibers and, up to normalization, provides an algebraic model of the quotient; cf. [7, Prop. 3.5]. This should be compared with the construction of the MRC fibration or the nef reduction.

6.2. Definable analyticity of q . Suppose that $q: \overline{X}(\mathbb{C}) \rightarrow Z := \overline{X}(\mathbb{C})/R$ is a map of definable topological spaces and the restriction $q: X^{\leq i-1} \rightarrow Z^{\leq i-1}$ is algebraic. Cover $Z^i \subset Z^{\leq i}$ by definable open sets $\{U_k\}$ with $U_k \subset Z^{\leq i}$. We construct a definable analytic morphism $\varphi_k: q^{-1}(U_k) \rightarrow \mathcal{B}_i$ with compact fibers whose Stein factorization coincides topologically with q , thus inducing a definable analytic structure to the atlas $\{U_k\}$ and the morphism q .

The morphism φ_k is obtained by the lift $\rho(X^i)$ of the quasiperiod map $\rho_{\underline{X}^i}$ on X^i constructed in §4.3, supplemented by the global sections of the Hodge bundle on $\overline{X}$, embedding $Z^{\leq i-1}$ in $\mathbb{P}(H_{\text{van}}^0(X^{\leq i-1}, L^k))$. The morphism φ_k is definable analytic as period maps are, and has compact fibers by Lemma 6.8. In addition, there we show that the former Hodge-theoretic set of sections separates points on X^i , while the latter algebro-geometric set of sections separates points on $X^{\leq i} \setminus X^i$ and vanishes along X^i by construction. Therefore,

the morphism φ_k is proper and the connected components of its fibers are isomorphic to the fibers of q .

Recall that the quasiperiod map $\rho(X^i)$ sits in the following diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{T}}(X^i) & \xrightarrow{\rho(X^i)} & \mathbb{P}(V_{i,\mathbb{C},x}^{\min}) \\ \uparrow & & \uparrow \\ X^i & \xleftarrow{\pi} \widetilde{X}^i & \xrightarrow{\rho_{X^i}} \mathbb{P}(V_{i,\mathbb{C},x}^{\text{tr}}) \end{array}$$

Lemma 6.8. *If V has neat monodromy, and L is integrable with torsion combinatorial monodromy, then the connected components of the fibers $\widetilde{F}$ of ρ_{X^i} , with $x \in X^i$, are isomorphic to the fibers of q , so compact.*

Proof. Since ρ_{X^i} is definable analytic, and X^i algebraic, $\widetilde{F} = \pi^{-1}(F)$ for some algebraic subset $F \subset X^i$ by definable Chow. Since $L|_F$ is flat, by integrability and neatness, the VHS $V_{X^i}^{\text{tr}}$ are trivial on any irreducible component of F , and so $V_{X^i}^{\min}$ is by Lemma 4.5. Since, by construction, any curve, intersecting X^i , along which $V_{X^i}^{\text{tr}}$ is trivial, is entirely contained in X^i , the closure of F in $\overline{X}$ is still contained in X^i , and actually it is proper, and F is a full fiber of q . Since $V_{X^i}^{\min}$ is trivial along F , the cover π on F is trivial, and $\widetilde{F}$ is isomorphic to a disjoint union of fibers of q . $\square$

7. COMPACT MODULI SPACES OF CALABI–YAU VARIETIES

According to the abundance conjecture, algebraic varieties should decompose into three fundamental building blocks: Fano varieties, Calabi–Yau varieties, and canonically polarized varieties. A fundamental goal in algebraic geometry is to construct well behaved parameter spaces of these building blocks called *moduli spaces*. The parametrizing spaces are themselves algebraic varieties (or stacks). To fully leverage the tools of algebraic geometry, it is desirable to compactify these spaces in a way that preserves the modular nature of the parametrization, so that boundary points retain meaningful information about the degenerations of the algebraic varieties being parametrized, or of their cohomological data. Advances in the moduli theory for canonically polarized varieties and for Fano varieties have been especially striking, culminating in two essentially definitive treatises [21, 33]. In contrast, many fundamental questions in the Calabi–Yau context remain open. An application of the b-semiampleness conjecture (Theorem 5.8) is the construction of a long-desired Hodge theoretic compactification of the moduli space of Calabi–Yau varieties, as we explain below. These compactifications are referred to as *non-classical or generalized Baily–Borel compactifications*, as they extend the classical Baily–Borel compactification of the moduli spaces of abelian and hyperkähler varieties; see [5].

Let $(\mathcal{M}, \lambda)$ be a moduli stack of polarized Calabi–Yau manifolds of dimension n with Hodge bundle λ . It is a Deligne–Mumford stack, so it admits a coarse moduli space M and a power of λ descends to M , say $\lambda_M^{(k)}$.⁶ The line bundle $\lambda_M^{(k)}$ is strictly nef, since no curve in M parametrizes an isotrivial family of Calabi–Yau manifolds.⁷

⁶The notation alludes to the fact that λ descends as a $\mathbb{Q}$ -line bundle of M , so $\lambda_M^{(k)}$ may not be the k -power of a line bundle on M .

⁷Technically, the moduli problem grants only that no curve in M parametrizes isotrivial family of *polarized* Calabi–Yau manifolds. However, even forgetting the datum of the polarization, the underlying family of varieties can never be isotrivial; see [8, §7.3.(ii)].

Theorem 7.1. *The coarse moduli space M admits a normal compactification M^{BBH} such that:*

- (1) $\lambda_M^{(k)}$ extends to an ample line bundle $\lambda_{M^{\text{BBH}}}^{(k)}$ on M^{BBH} for k sufficiently divisible;
- (2) (functoriality or minimality of the compactification) for any family $\bar{f}: \bar{Y} \rightarrow \bar{X}$ of polarized Calabi–Yau manifolds, parametrized by $\mathcal{M}$, with $K_{\bar{Y}/\bar{X}} \sim_{\bar{f}} 0$, snc discriminant D , dlt in codimension one (or alternatively, locally stable over $(\bar{X}, D)$) and Hodge bundle $\lambda_{\bar{X}}$, the classifying map $\varphi_f: X = \bar{X} \setminus D \rightarrow \mathcal{M} \rightarrow M$ extends to a regular morphism $\varphi_{\bar{f}}: \bar{X} \rightarrow M^{\text{BBH}}$ compatibly with the Hodge bundle, i.e., with $\varphi_{\bar{f}}^* \lambda_{M^{\text{BBH}}}^{(k)} \simeq \lambda_{\bar{X}}^k = (\bar{f}_* \omega_{\bar{Y}/\bar{X}})^{\otimes k}$.

$$\begin{array}{ccc} X & \xrightarrow{\quad} & \bar{X} \\ \downarrow \varphi_f & & \downarrow \varphi_{\bar{f}} \\ \mathcal{M} & \longrightarrow & M \hookrightarrow M^{\text{BBH}}. \end{array}$$

Sketch of the proof. If $X \rightarrow M$ is surjective with connected fibers, set

$$(7.1) \quad M^{\text{BBH}} = \text{Proj} \bigoplus_i H^0(\bar{X}, \lambda_X^i).$$

The ring $\bigoplus_i H^0(\bar{X}, \lambda_X^i)$ is finitely generated by the semiampleness of $\lambda_{\bar{X}}$, and M is an open subset of M^{BBH} , since $\lambda_M^{(k)}$ is strictly nef. $\square$

Remark 7.2. We can rewrite (7.1), independently of the choice of $\bar{X}$, as the proj of the graded ring of sections of the Hodge bundle on M with moderate growth, i.e., whose Hodge norm grows sub-polynomially at the boundary:

$$M^{\text{BBH}} = \text{Proj} \bigoplus_i H_{\text{mg}}^0(M, \lambda_M^{(ki)}).$$

In fact, the moderate growth hypothesis is unnecessary in most cases (when the Baily–Borel boundary ends up codimension ≥ 2). This generalizes the construction of the moduli space of abelian varieties via automorphic forms.

Remark 7.3. Theorem 7.1 holds more generally for moduli space of polarized klt log Calabi–Yau pairs, provided that we replace M with its normalization. The coarse moduli space of polarized Calabi–Yau manifolds is automatically normal (with only quotient singularities) by the unobstructedness of their deformations.

7.1. Boundary of the compactification. The compactification of the moduli space of abelian varieties $\mathcal{A}_g$ in Theorem 7.1 recovers the usual Baily–Borel compactification

$$\mathcal{A}_g^{\text{BB}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \dots \sqcup \text{pts.}$$

The boundary points parametrize abelian varieties of lower dimension. Given a degeneration of abelian varieties of dimension g over a punctured disk, up to finite cover, it admits a filling with a semiabelian variety extension of an abelian variety of dimension $\geq g$ and an algebraic torus. The boundary points record the compact part of the semiabelian variety. Alternatively from a birational viewpoint, up to a finite cover, the family admits a minimal snc filling whose minimal stratum (or source) is an abelian variety corresponding to the boundary point. In general, there is a stratification

$$M^{\text{BBH}} = M \sqcup M_1 \sqcup \dots \sqcup M_r$$

where informally M_i is a moduli space of sources of degenerations, up to finite ambiguity. The exact moduli interpretation however is unclear. By the proof, M_i corresponds to the

strata Z_i , which up to finite ambiguity are images of period maps of V_Σ^{tr} . Note that V_Σ^{tr} is a rational VHS admitting an integral structure, but in fact the integral structures glue along the strata of X^i . Moreover, in this geometric case, there is a global polarization on X^i , hence there is a global period map. The connection between this integral structure and the integral structure on the transcendental lattice of the source is not clear, however, since the identification implicitly uses Hard Lefschetz with respect to the local monodromy. Thus, while the crepant birational isomorphism class of the source is constant along the fibers of $X^i \rightarrow M_i$, it seems unlikely that the map in general is the quotient by this equivalence relation.

Example 7.4. Let $\pi: \tilde{\mathcal{A}}_2 \rightarrow \mathcal{A}_2^{\text{BB}}$ be a log resolution and choose a general complete intersection surface Z in $\tilde{\mathcal{A}}_2$. In general, the intersection of Z with a divisor in $\tilde{\mathcal{A}}_2$ dominating $\mathcal{A}_1 \subset \mathcal{A}_2^{\text{BB}}$ will be generically a multisection. We have that $(Z \cap \mathcal{A}_2)^{\text{BB}}$ is the normalization of $\pi(Z)$, i.e., Z itself, and admits a boundary stratum Z_Σ whose period map $Z_\Sigma \rightarrow \mathcal{A}_1$ can have arbitrary high degree, according to the degrees of the ample line bundles cutting Z .

Remark 7.5. The boundary of M^{BBH} can have very large codimension. For instance, if $\mathcal{M}$ parametrises Calabi–Yau threefolds, then

$$M^{\text{BBH}} = M \sqcup M_{\text{K3 surfaces}} \sqcup M_{\text{abelian surfaces}} \sqcup M_{\text{elliptic curves}} \sqcup \text{pts.}$$

The dimension of any boundary stratum is ≤ 19 (which is the largest possible dimension for moduli of Calabi–Yau varieties of dimension ≥ 2), but the dimension of M is $h^{1,2}$ of the Calabi–Yau threefold, e.g., 101 for the quintic 3-fold.

Remark 7.6. The boundary of the moduli space of canonically polarized or K-stable Fano manifolds parametrises degenerate varieties with ample or anti-ample canonical bundle of the *same* dimension. In general, in the Calabi–Yau case, the boundary points parametrises varieties of lower dimension, which obstructs the possibility of defining a naive moduli functor of flat families of degenerate Calabi–Yau varieties.

7.2. Compactification via Griffiths bundle. The semiampness of the Griffiths bundle (Theorem 5.12) grants the existence of a similar compactification M^{BB} where now the Griffiths bundle extends amply. Since the Griffiths bundle is the sum of the Hodge bundle and a semipositive line bundle, the Hodge bundle vanishes along all curves where the Griffiths bundle vanishes, thus inducing a morphism $M^{\text{BB}} \rightarrow M^{\text{BBH}}$. The morphism can have positive dimensional fibers. In particular, it contracts curves along which the limiting mixed Hodge structure of a degeneration has moduli but the associated source has no moduli.

Example 7.7 (Quintic threefolds). Let M the moduli space of smooth quintic threefolds. M^{BB} contains a boundary stratum parametrizing snc varieties $H+Q$, union of a hyperplane H and a quartic threefold Q . By the Clemens–Schmid sequence, the primitive cohomology $H^3(Q, \mathbb{Z})^{\text{prim}}$, with Hodge numbers $(0, 30, 30, 0)$, is a sub Hodge structure of the third graded piece of the limiting mixed Hodge structure, corresponding to the middle row $(0, 81, 81, 0)$ in the diamond in Example 5.11. The generic Torelli theorem for quartic threefolds [14] forces the boundary stratum to have dimension greater than the dimension of the moduli space of quartic threefold, i.e., ≥ 45 . On the other hand, the dimension of the boundary of M^{BB} is ≤ 19 by Remark 7.5, so $M^{\text{BB}} \rightarrow M^{\text{BBH}}$ contracts curves in the boundary of M^{BB} .

It is even possible to contract curve along which the generic period map extends, i.e., where the limiting mixed Hodge structure is pure. Indeed, there exists a family of quintic threefold on $\mathbb{A}_{x,y}^2$ such that

- (1) along the y -axis, it is a trivial family of a fixed quintic threefold with a unique isolated singularity;
- (2) the vanishing cycle of the smoothing along horizontal directions is the Hodge structure of an elliptic curve with varying j -invariant as we vary the point on the y -axis.

The Griffiths bundle is not flat along the y -axis since it records the moving VHS of the vanishing cycle. On the other hand, the Hodge bundle is flat along the y -axis since there the family is trivial.

7.3. A metric interpretation. The compact moduli of canonically polarized and Fano varieties admits a metric characterization as Gromov–Hausdorff compactification at volume scale of moduli spaces of smooth Kähler–Einstein canonically polarized and Fano varieties. Spotti has conjectured a relation between M^{BBH} and Gromov–Hausdorff compactification of Ricci–flat Calabi–Yau varieties at volume scale. See [27, §4].

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