Mixed non-abelian Hodge theory and the linear Shafarevich conjecture

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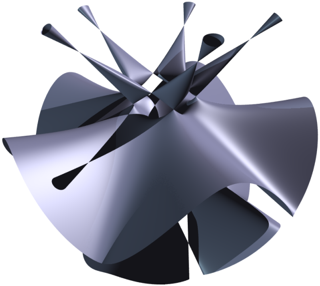
Topology of algebraic varieties

Let X be a complex algebraic variety.



How do the topological and algebraic structures interact?

We should at least restrict to **normal** algebraic varieties...

Even stronger control for **smooth** and **compact (or projective)**.

Classical Hodge theory for smooth projective X

$$\underset{\text{Betti}}{H^k(X,\mathbb{C})} \quad \underset{\substack{\uparrow \\ \textit{integration}}}{\cong} \quad \underset{\text{de Rham}}{H^k(X,\Omega^\bullet_X)} \quad \underset{\substack{\uparrow \\ \textit{represent by} \\ \textit{harmonic form} \\ \textit{then decompose}}}{\cong} \quad \underset{\text{Dolbeault}}{\bigoplus_{p+q=k} H^q(X,\Omega^p_X)}$$

Example: For any $a \in H^1(X,\mathbb{C})$,

$$\begin{aligned} a(\gamma) &= \int_\gamma [\alpha] \\ &= \int_\gamma \underset{\substack{\uparrow \\ \textit{holomorphic} \\ \textit{1-form}}}{\alpha^{1,0}} + \underset{\substack{\uparrow \\ \textit{anti-holomorphic} \\ \textit{1-form}}}{\alpha^{0,1}} \end{aligned}$$

This is packaged into a "pure Hodge structure."

Non-abelian Hodge theory for smooth projective X

Deligne (70s), Corlette, Donaldson, Hitchin (80s), Simpson (90s), etc.: instead consider **representations of $\boldsymbol{\pi_1(X)}$** $(= H^1(X, \mathrm{GL}_r(\mathbb{C})))$

$$M_B(X,r) \quad \cong \quad M_{dR}(X,r) \quad \cong \quad M_{Dol}(X,r)$$

$$\left\{\begin{array}{c}\text{rank } r\\ \text{semisimple}\\ {}_{\pi_1}\ \text{representations}\end{array}\right\} \qquad \left\{\begin{array}{c}\text{rank } r\\ \text{semisimple}\\ \text{flat vector bundles}\\ (E,\nabla)\end{array}\right\} \qquad \left\{\begin{array}{c}\text{rank } r\\ \text{polystable}\\ \text{Higgs bundles}\\ (E,\theta)\end{array}\right\}$$

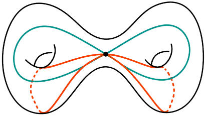
integrating connection

complex-analytic

equip (E,∇) with unique harmonic metric

real-analytic

Example:



$$\pi_1(X) = \left\langle a_1, a_2, b_1, b_2 \;\middle|\; \prod_i [a_i, b_i] = 1 \right\rangle$$

$$M_B(X,r)$$

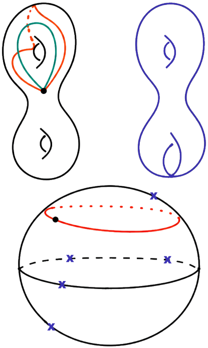
$$||$$

$$\mathrm{GL}_r(\mathbb{C}) \Big\backslash \left\{ A_1, A_2, B_1, B_2 \in \mathrm{GL}_r(\mathbb{C}) \;\middle|\; \prod_i [A_i, B_i] = 1 \right\}$$

Variations of Hodge structures (VHSs)

= representations that “look like” the monodromy of a family of algebraic varieties.

= monodromy representations underlying families of Hodge structures.



$y^2 = x(x^3-1)(x-t)$

$$\begin{tikzcd} M_B(X,r) \arrow[r, phantom, "\cong"] & M_{dR}(X,r) \arrow[r, phantom, "\cong"] & M_{Dol}(X,r) \arrow[loop above, "\mathbb{C}^*"] \end{tikzcd}$$

Simpson: $\mathbb{C}^*$-fixed points are VHSs.

$\mathbb{C}^*$-limits exist as $\lambda \to 0$.

$\Rightarrow$ Every representation can be deformed to a VHS.

Emerging theme: all representations controlled by VHSs!

From smooth projective X to normal X: the weight one part

Deligne ('70s) equips $H^k(X,\mathbb{C})$ with a **mixed** Hodge structure
$=$ filtration $W_\bullet H^k(X,\mathbb{C})$ whose graded pieces are pure Hodge structures.

$$\begin{tikzcd}
0 \arrow[r] & W_1H^1(X,\mathbb{C}) \arrow[r] \arrow[d, phantom, "\cap"] & W_2H^1(X,\mathbb{C}) \arrow[r] \arrow[d, phantom, "||"] & \operatorname{gr}_2^W H^1(X,\mathbb{C}) \arrow[r] \arrow[d, phantom, "||"] & 0 \\
& H^1(\overline{X'},\mathbb{C}) & H^1(X,\mathbb{C}) & \begin{array}{c}\textit{residues of 1-forms which}\\ \textit{extend with log poles on } \overline{X'}\end{array} &
\end{tikzcd}$$

Theorem 1 (Simpson, T. Mochizuki, B–Brunebarbe–Tsimerman, Tran, '90-'26)

$$\begin{tikzcd}
M_B^{\mathrm{unip}}(X,r) & \cong & M_{dR}^{\mathrm{nilp}}(X,r) & \cong & M_{Dol}^{\mathrm{nilp}}(X,r) \arrow[loop above, "\mathbb{C}^*"] \\
\left\{\begin{array}{c}\textit{rank } r\\ \textit{semisimple}\\ \pi_1 \textit{ representations}\\ \textit{with unipotent}\\ \textit{local monodromy}\end{array}\right\} & & \left\{\begin{array}{c}\textit{rank } r\\ \textit{semisimple}\\ (E,\nabla)\\ \textit{extending}\\ \textit{logarithmically}\\ \textit{with nilpotent}\\ \textit{residues}\end{array}\right\} & & \left\{\begin{array}{c}\textit{rank } r\\ \textit{polystable}\\ (E,\theta)\\ \textit{extending}\\ \textit{logarithmically}\\ \textit{with nilpotent}\\ \textit{residues}\end{array}\right\} \\
& \textit{complex-analytic} \arrow[uu] & & \textit{continuous} \arrow[uu] &
\end{tikzcd}$$

What about the higher weight part?

Simpson: Think about the harmonic correspondence infinitesimally.

Theorem 2 (Simpson–Eyssidieux '11, B–Brunebarbe–Tsimerman '24)

Deformation rings of points of $M_B(X, r)$ are equipped with **mixed twistor structures**; universal families are **variations of mixed twistor structures**.

What is a mixed twistor structure?

A	$\mathbb{C}^*$-fixed	point of $M_B(X, r)$	is a	VHS
	$\mathbb{C}^*$-equivariant	(mixed) twistor structure		(mixed) Hodge structure

Idea: Harmonic metrics ⇒ **any** semisimple representation underlies a variation of pure twistor structures.

Deformation theory of V governed by $H^i(X, End(V))$; these are equipped with mixed twistor structures (Simpson, Sabbah, Mochizuki).

VHSs control all representations

Corollaries

(of Thm 2) Completions of VHS points of $M_B(X, r)$ come from VHSs.

(of Thm 1) Every component of $M_B^{\mathrm{unip}}(X, r)$ contains a VHS point.

Theorem 3 (Esnault–Kerz '21, B–Brunebarbe–Tsimerman '24)

Quasiunipotent local monodromy points of $M_B(X, r)$ are Zariski dense.

Idea: Eigenvalues of local monodromy / residues are $\overline{\mathbb{Q}}$-bialgebraic with respect to the exponential map. Now use transcendence theory.

Theorem 4 (B–Brunebarbe–Tsimerman '24)

VHSs are Zariski dense in $\mathcal{M}_B(X, r)$.

- Allows us to see **all** representations—not just semisimple ones.
- Harder! Must use formal twistor geometry package from Thm 2.

Linear Shafarevich morphisms and the Griffiths conjecture

Given a normal X with $\rho : \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$:

- How many positive-dimensional $Z \subset X$ are there with $\rho|_{\pi_1(Z)}$ is finite?
- Is there an open embedding $X \subset \check{X}$ s.t. ρ factors through $\pi_1(\check{X})$?

Theorem 5 (B–Brunebarbe–Tsimerman ’24)

There is a unique algebraic $f_\rho : X \to Y$ with connected general fiber s.t.:

- ρ factors through $\pi_1(Y)$.
- For every positive-dimensional $Z \subset Y$, $\rho|_{\pi_1(Z)}$ has infinite image.
- There is no open embedding $Y \subset \check{Y}$ through which ρ factors.

Y is a sort of “minimal model” of ρ.

Previous work by Kollár, Campana, Katzarkov (90s), Eyssidieux (’04), Brunebarbe (’23), Deng–Yamanoi (’23).

Linear Shafarevich morphisms and the Griffiths conjecture

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- If ρ comes from a $\mathbb{Z}$-VHS, f_ρ is analytically given by the **period map:** the map $X \to M = \Gamma\backslash\mathbb{D}$ to the moduli space of Hodge structures.
  - Griffiths (ICM 1970) conjectured period maps are algebraic.
  - Proven by BBT '23 using **o-minimal GAGA** (also BBT '23).
  - **o-minimal GAGA** gives algebraization results for analytic structures on **noncompact** spaces...if they are **tame** in a model-theoretic sense.
- Now use density of VHSs + o-minimal GAGA + $\cdots$
  - Still have period maps $\widetilde{X} \to \mathbb{D}$ upstairs...with some work this is enough!

Linear Shafarevich conjecture

Theorem 6 (B–Brunebarbe–Tsimerman '24)

For a "minimal model" Y of a representation ρ, the universal cover $\widetilde{Y}$ is Stein.

This means $\widetilde{Y}$ has a rich holomorphic function theory.

Linear Shafarevich Conjecture

(Eyssidieux–Katzarkov–Pantev–Ramachandran '12, B–Brunebarbe–Tsimerman '24)

If $\pi_1(X)$ admits a faithful representation ρ, then the universal cover $\widetilde{X}$ is quasi-holomorphically convex.

Linear Shafarevich conjecture

Theorem 6 (B–Brunebarbe–Tsimerman '24)

For a "minimal model" Y of a representation ρ, the universal cover $\widetilde{Y}$ is Stein.

So far I've skipped a parallel story at the non-archimedean places…

harmonic	archimedean metrics	relate	$\mathbb{C}$-local systems	to	VHSs
	non-archimedean norms		$\mathbb{Q}_p$-local systems		$\mathbb{Z}_p$-local systems

- Harmonic non-archimedean norms exist by Gromov–Schoen, Jost–Zuo (90s), Corlette–Simpson ('08), Brotbek–Daskalopoulos–Deng–Mese ('22) $+\epsilon$.
- Putting harmonic maps at every place v together we get a **proper** map:

$$Y \to \Gamma \backslash \prod_v \Delta_v$$

- Use this to produce a strictly plurisubharmonic compact exhaustion.
  - **Need:** norms compatible with standard functors (e.g. nearby cycles).
  - **Also:** strict positivity of current coming from non-archimedean factors.

Thanks!