ERRATUM TO "p-TORSION MONODROMY REPRESENTATIONS OF ELLIPTIC CURVES OVER GEOMETRIC FUNCTION FIELDS"

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Abstract. We correct two errors in [3]—in fact, both errors are erroneous simplifications of correct arguments that appeared in the original preprint [2]. We supply these arguments here. The main result is unaffected.

1. **Summary of errors.** The modular surface $Z(p) := \mathrm{PSL}_2(\mathbb{F}_p)\backslash X(p) \times X(p)$ obtained by taking the quotient of the product $X(p) \times X(p)$ of (compactified) full level modular curves by the diagonal action on the framing parametrizes triples (E_1, E_2, ϕ) where E_1, E_2 is a pair of elliptic curves and $\phi : E_1[p] \xrightarrow{\cong} E_2[p]$ an isomorphism of their p-torsion. The surface $Z(p)$ naturally contains Hecke curves for which the isomorphism is induced by an isogeny. The main theorem of [3] is that for any d, and for sufficiently large p (depending on d) any curve of gonality d on $Z(p)$ is a Hecke curve.

The proof of the main theorem proceeds by passing to the associated rational curve $\mathbb{P}^1 \to \mathrm{Sym}^d Z(p)$, and then to the cover $\pi : C \to \mathbb{P}^1$ which admits a map $C \to (X(p) \times X(p))^d$. It is then shown that the incidence of C along the ramification locus of the quotient $(X(p) \times X(p))^d \to \mathrm{Sym}^d Z(p)$ is small compared to the volume provided C is not a Hecke curve. There are four kinds of ramification loci: ramification arising from the quotient $X(p) \times X(p) \to Z(p)$, and ramification arising from the quotient $Z(p)^d \to \mathrm{Sym}^d Z(p)$. The former consists of Heegner and anti-Heegner CM points on $X(p) \times X(p)$, which live above the points with j-invariants $(0, 0)$ and $(1728, 1728)$ of $X(1) \times X(1)$, and singular bicusps of $X(p) \times X(p)$, which live above the cusp (∞, ∞); the latter is reduced to the diagonal in $(X(p) \times X(p))^2$. We denote by CM^+ (resp. CM^-) the set of Heegner (resp. anti-Heegner) CM points and by SBC the set of singular bicusps.

There are two errors:

- [3, Prop. 14] states that CM points of the same type with the same projection to $X(1) \times X(1)$ which are close to each other lie on a low-degree Hecke curve. As anti-Heegner CM points cannot lie on Hecke curves, this in particular implies anti-Heegner CM points always repel (see [3, Remark 16]). This is false: the real analytic involution which conjugates the second factor preserves the hyperbolic metric and swaps Heegner and anti-Heegner CM points. Close anti–Hegner CM points will however lie on low degree conjugate Hecke curves, and this is what was used in the original argument (as well as in [1]).
- The proof of [3, Prop. 26] concludes with $\mathrm{mult}_{\mathrm{SBC}}(C) \ll p^{-1+\delta} \mathrm{Deg}(C)$, but we need $\mathrm{mult}_{\mathrm{SBC}}(C) \ll p^{-1-\delta} \mathrm{Deg}(C)$. A slightly more careful analysis of singular bicusps lying on low degree Hecke curves must be used, which again appears in the original preprint.

2. **Repulsion of CM points.** The correct version of [3, Prop. 14] is the same statement, but only for Heegner CM points.

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Proposition 2.1 (cf. [2, Prop. 15], [1, Prop. 8])**.** *For all sufficiently small $\delta > 0$ and sufficientlty large p we have the following. Let $\xi, \xi' \in \mathrm{CM}^+$ be distinct Heegner CM points in $X(p) \times X(p)$ with the same projections to $X(1) \times X(1)$ such that $B(\xi, \delta\rho_{X(p)}) \cap B(\xi', \delta\rho_{X(p)}) \neq \varnothing$. Then ξ and ξ' lie on a Hecke divisor T_m with $m = p^{O(\delta)}$.*

The proof is the same taking $h_0 = 1$ (which is the Heegner CM point case). Let $\eta \in X(p)$ be the image of $i \in \mathbb{H}$ or $e^{i\pi/3} \in \mathbb{H}$, H its stabilizer in $\mathrm{PSL}_2(\mathbb{F}_p)$. In general there does not exist $h_0 \in \mathrm{SL}_2(\mathbb{Z})$ reducing to the non-identity coset of H in the normalizer $N(H)$, and this is the error.

3. **Relative volume estimates for the conjugate diagonal.** As described above, to handle anti-Heegner CM points we need a bound on the volume acquired by a curve near the conjugate diagonal. Recall that we denote by $a(r)$ the area of a 1-dimensional hyperbolic disk of radius r; normalizing the metric to have holomorphic sectional curvature -1, we have $a(r) = 4\pi \sinh^2(r/2)$.

Proposition 3.1 (cf. [2, Prop. 26], [1, Prop. 11])**.** *Let X be a compact hyperbolic complex curve with constant holomorphic sectional curvature, $\bar{\Delta} \subset X \times \bar{X}$ the conjugate diagonal, and $C \subset X \times \bar{X}$ a complex curve. Then for $r < \rho_X/2$,*

$$\frac{1}{a(2r)^{1/2}} \operatorname{vol}(C \cap B(\bar{\Delta}, r))$$

is an increasing function of r.

Proof. The proof is very similar to [3, Prop. 23]. Consider the function ψ on $\mathbb{D} \times \mathbb{D}$ given by

$$\psi(z, w) = \tanh^2(d_{\mathbb{D}}(z, \bar{w})/2) = \left| \frac{\bar{w} - z}{1 - zw} \right|^2.$$

ψ is invariant under the conjugate diagonal action and descends to $B(\bar{\Delta}, r)$ provided $r < \rho_X/2$. For any function $h : \mathbb{R} \to \mathbb{R}$, we compute that at $(0, w)$, the potential $h \circ \psi$ yields a form

$$i\partial\bar{\partial}(h \circ \psi) = h'(\psi) \begin{pmatrix} (1 - |w|^2)^2 & w^2 \\ \bar{w}^2 & 1 \end{pmatrix} + h''(\psi) \begin{pmatrix} |w|^2(1 - |w|^2)^2 & -w^2(1 - |w|^2) \\ -\bar{w}^2(1 - |w|^2) & |w|^2 \end{pmatrix}.$$

Taking $f(s) = -\log(1 - s)$, for instance, we have by direct computation

$$i\partial\bar{\partial}(f \circ \psi) = \begin{pmatrix} 1 & 0 \\ 0 & (1 - |w|^2)^{-2} \end{pmatrix} = \frac{\omega_{\mathbb{D} \times \mathbb{D}}}{2}.$$

Note that $f(\tanh^2(d/2)) = \log\cosh^2(d/2)$. Taking $g(s) = 2\sin^{-1}\sqrt{1 - e^{-s}}$ then $(g \circ f)(s) = 2\sin^{-1}\sqrt{s}$ and we compute

$$(g \circ f)'(s) = \frac{1}{s^{1/2}(1 - s)^{1/2}}$$
$$(g \circ f)''(s) = \frac{2s - 1}{2s^{3/2}(1 - s)^{3/2}}$$

which by the above yields

$$i\partial\bar{\partial}(g \circ f \circ \psi) \geq 0.$$

By Stokes' theorem,

$$\begin{aligned}\operatorname{vol}(C \cap B(\bar{\Delta}, r)) &= 2\int_{C\cap B(\bar{\Delta},r)} i\partial\bar{\partial}(f\circ\psi)\\ &= \frac{2}{g'(\log\cosh^2(r))}\int_{C\cap B(\bar{\Delta},r)} i\partial\bar{\partial}(g\circ f\circ\psi)\\ &= 2\sinh(r)\int_{C\cap B(\bar{\Delta},r)} i\partial\bar{\partial}(g\circ f\circ\psi)\end{aligned}$$

and therefore $\frac{1}{\sinh(r)}\operatorname{vol}(C\cap B(\bar{\Delta}, r))$ is an increasing function. □

4. **Multiplicity estimates.** Recall that we denote by $\operatorname{Deg}(C)$ the degree of $C \subset X(p) \times X(p)$ with respect to the canonical bundle, which up to an absolute constant is equal to $\operatorname{vol}(C)$.

4.1. *Anti-Heegner CM points.* [3, Prop. 25] is still correct, but the proof as written only applies to Heegner CM points. We use Proposition 3.1 (as in [1]) instead:

Proposition 4.1 (cf. [2, Prop. 28], [1, Prop. 13])**.** *For all sufficiently small $\delta > 0$, all sufficiently large p, and for any non-Hecke curve $C \subset X(p) \times X(p)$,*

$$\operatorname{mult}_{\mathrm{CM}^-}(C) = O(p^{-\delta}\operatorname{Deg}(C)).$$

Proof. Take $d = p^{\delta_1}$. We would like to perform the same trick as in [3, Prop. 25] for the anti-Heegner CM points, and we again partition the points of CM^- into

$$T = \mathrm{CM}^- \cap \cup_{m<d}\bar{T}_m \quad \text{and} \quad S = \mathrm{CM}^- - T$$

where $\bar{\cdot}$ denotes complex conjugation on the second factor. For sufficiently small $\delta_2 > 0$, it will still be the case that balls of radius $\delta_2\rho_{X(p)}$ around points of S are disjoint, but the multiplicity of a curve C along $\bar{T}_m$ doesn't make sense, so we adjust the argument slightly:

$$\begin{aligned}\operatorname{mult}_{\mathrm{CM}^-}(C) &= \operatorname{mult}_S(C) + \operatorname{mult}_T(C)\\ &\ll \frac{1}{a(\delta_2\rho_{X(p)})}\sum_{\xi\in S}\operatorname{vol}(C\cap B(\xi,\delta_2\rho_{X(p)})) + \sum_{m<d}\operatorname{mult}_{\mathrm{CM}^-\cap\bar{T}_m}(C)\\ &\ll p^{-2\delta_2}\operatorname{Deg}(C) + \sum_{m<d}\operatorname{mult}_{\mathrm{CM}^-\cap\bar{\Delta}}(T_m^*C). \qquad (1.1)\end{aligned}$$

Balls of a fixed small radius $\epsilon > 0$ ($\epsilon = \frac{1}{2}d_{X(1)_p}(q_2, q_3)$ is sufficient) around points in $\mathrm{CM}^- \cap \bar{\Delta}$ are disjoint, and therefore by the theorem of Hwang–To [3, Thm. 19] we have

$$\begin{aligned}\operatorname{mult}_{\mathrm{CM}^-\cap\bar{\Delta}}(T_m^*C) &\ll \frac{1}{a(\epsilon)}\sum_{\xi\in\mathrm{CM}^-\cap\bar{\Delta}}\operatorname{vol}(T_m^*C\cap B(\xi,\epsilon))\\ &\ll \operatorname{vol}(T_m^*C\cap B(\bar{\Delta},\epsilon))\\ &\ll \frac{\sinh(\rho_{X(p)}/2)}{\sinh(\epsilon)}\operatorname{vol}(T_m^*C)\end{aligned}$$

where we've used Proposition 3.1 (and [3, Cor. 8]) in the last step. Combining this with equation (1.1), we have

$$\operatorname{mult}_{\mathrm{CM}^-}(C) \ll (p^{-2\delta_2} + d^3p^{-1})\operatorname{Deg}(C)$$

and the result follows. □

4.2. *Singular bicusps.* The statement of [3, Prop. 26] is correct, but the proof is incomplete.

Proposition 4.2 (cf. [2, Prop. 29]). *For all sufficiently small $\delta > 0$, all sufficiently large p, and for any non-Hecke curve $C \subset X(p) \times X(p)$,*

$$p\,\mathrm{mult}_{\mathrm{SBC}}(C) = O(p^{-\delta}\,\mathrm{Deg}(C)).$$

Proof. Take $d = p^{\delta_1}$, and again partition the points of SBC into

$$T = \mathrm{SBC} \cap \cup_{m<d} T_m \quad \text{and} \quad S = \mathrm{SBC} - T.$$

By [3, Prop. 17], for δ_2 a sufficiently small multiple of δ_1, any point in $X(p) \times X(p)$ is in at most two of the balls $B(\xi, (1/2+\delta_2)\rho_{X(p)})$ for $\xi \in S$. By [3, Thm. 19(a)], for each $\xi \in \mathrm{SBC}$,

$$\mathrm{mult}_\xi(C) \ll \frac{1}{a((1/2+\delta_2)\rho_{X(p)})}\,\mathrm{vol}(C \cap B(\xi, (1/2+\delta_2)\rho_{X(p)})))$$

and it therefore follows that

$$\begin{aligned} \mathrm{mult}_S(C) &\ll \frac{\mathrm{vol}(C)}{a((1/2+\delta_2)\rho_{X(p)})} \\ &\ll p^{-1-2\delta_2}\,\mathrm{Deg}(C). \end{aligned} \tag{1.2}$$

Now for any $m < d$, and sufficiently small $\delta_3 > 0$ (independent of δ_1 and δ_2),

$$\begin{aligned} \mathrm{mult}_{\mathrm{SBC}\cap T_m}(C) &= \sum_{\xi \in \mathrm{SBC}\cap T_m} \mathrm{mult}_\xi(C) \\ &\ll \sum_{\xi \in \Delta\cap\mathrm{SBC}} \mathrm{mult}_\xi(T_m^* C) \\ &\ll \frac{1}{a((1/2+\delta_3)\rho_{X(p)})} \sum_{\xi \in \Delta\cap\mathrm{SBC}} \mathrm{vol}(T_m^* C \cap B(\xi, (1/2+\delta_3)\rho_{X(p)})). \end{aligned}$$

By [3, Prop. 12(b)], any point within $(1/2+\delta_3)\rho_{X(p)}$ of two distinct singular bicusps on Δ must be within $\delta_3\rho_{X(p)} + O(1)$ of Δ, so by [3, Thm. 19(b)]

$$\begin{aligned} \mathrm{mult}_{\mathrm{SBC}\cap T_m}(C) &\ll \frac{d^2 \cdot \mathrm{vol}(C)}{a((1/2+\delta_3)\rho_{X(p)})} + \frac{\mathrm{vol}(T_m^* C \cap B(\Delta, \delta_3\rho_{X(p)} + O(1)))}{a((1/2+\delta_3)\rho_{X(p)})} \\ &\ll \frac{d^2 \cdot \mathrm{vol}(C)}{a((1/2+\delta_3)\rho_{X(p)})} \cdot \left(1 + \frac{\cosh(\delta_3\rho_{X(p)} + O(1))}{\cosh(\rho_{X(p)}/2)}\right) \\ &\ll p^{-1-2\delta_3+2\delta_1}\,\mathrm{Deg}(C) \end{aligned}$$

where the second inequality follows from [3, Prop. 23]. Thus,

$$\mathrm{mult}_T(C) \ll p^{-1-2\delta_3+3\delta_1}\,\mathrm{Deg}(C).$$

Combining this with equation (1.2) yields the claim.

□

5. **Finishing up.** All remaining statements and their proofs from [3] are correct as written.

References

[1] B. Bakker and J. Tsimerman. On the Frey-Mazur conjecture over low genus curves. `arXiv:1309.6568`, 2013.

[2] B. Bakker and J. Tsimerman. p-torsion monodromy representations of elliptic curves over geometric function fields. `arXiv:1403.7168v1`, 2014.

[3] B. Bakker and J. Tsimerman. p-torsion monodromy representations of elliptic curves over geometric function fields. *Ann. of Math. (2)*, 184(3):709–744, 2016.