Baily–Borel compactifications of period images and the b-semiampleness conjecture

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The classical story: $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$

(1) (Satake '56) $A_g^{\mathrm{SBB}} = A_g \sqcup A_{g-1} \sqcup \cdots \sqcup A_1 \sqcup A_0$.

(2) (Baily '58) $A_g^{\mathrm{SBB}} =$ Proj(graded ring of automorphic forms).

(3) <u>modular interpretation</u>.

$$\begin{tikzcd} Z^\circ \arrow[r, hook] \arrow[d] & Z \arrow[d] & Z_0 \arrow[l, hook'] \arrow[d] \\ C^\circ \arrow[r, hook] & C & 0 \arrow[l, phantom, "\ni"] \end{tikzcd}$$

- Z°/C° a family of abelian varieties
- Z/C is the (semistable) Neron model
- Identity comp. of Z_0 is a semi-abel. variety
- Compact part is limiting point in boundary

(4) <u>natural polarization</u>. $\pi : Z \to \mathcal{A}_g$ the universal family,
$L = \det \pi_* \Omega_{Z/\mathcal{A}_g}$ extends to an ample bundle $L_{A_g^{\mathrm{SBB}}}$ (up to a power).

The classical story: $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$

(5) universality. (Borel '72)

$$\begin{tikzcd} X^\circ := X\backslash D \arrow[r, hook] \arrow[d, "f"'] & X \arrow[d, dashed, "\overline{f}"] \\ A_g \arrow[r, hook] & A_g^{\mathrm{SBB}} \end{tikzcd}$$

- (X, D) log smooth
- $\overline{f}^* L_{A_g^{\mathrm{SBB}}} \cong L_X$

(5') Even have (5) in the analytic category, i.e. $(X, D) = (\Delta^k, \text{coordinate hyperplanes})$

(Satake '60, Baily–Borel '66) Generalized to arbitrary arithmetic locally symmetric varieties.

Period maps

Let (X, D) log smooth proper

$\pi : Z^\circ \to X^\circ$ be a smooth projective family,

$V = R^k\pi_*\mathbb{Z}_{(Z^\circ)^{\mathrm{an}}}$ equipped with its polarizable $\mathbb{Z}$-VHS (filtration $F^\bullet V$ on $\mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} V$).

$$\begin{tikzcd}
(X^\circ)^{\mathrm{an}} \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & \Gamma\backslash\mathbb{D} \\
& \mathcal{Y} := \overline{\operatorname{img}\varphi} \arrow[ur, hook] &
\end{tikzcd}$$

Question (Griffiths '70).

(A) Is $\mathcal{Y}$ algebraic?

(B) Is Griffiths bundle $L_{\mathcal{Y}} := \bigotimes_p \det F^pV$ algebraic? Ample?

(C) Is there a $\mathcal{Y}^{\mathrm{BB}}$?

Main theorem 1

Question (Griffiths '70).

(A) Is $\mathcal{Y}$ algebraic?

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Theorem (B–Brunebarbe–Tsimerman '23)

$\left((X^\circ)^{\mathrm{an}} \xrightarrow{\psi} \mathcal{Y}\right) = \left(X^\circ \xrightarrow{f} Y\right)^{\mathrm{an}}$ and $L_{\mathcal{Y}} = (L_Y)^{\mathrm{an}}$ all algebraic, L_Y ample.

(C) Is there a $\mathcal{Y}^{\mathrm{BB}}$?

Theorem 1 (B–Filipazzi–Mauri–Tsimerman)

- $B_Y := \bigoplus_k H^0_{mg}(Y, L_Y^k)$ *is finitely generated.*
- $Y^{\mathrm{BB}} := \operatorname{Proj} B_Y$ *is projective compactification of* Y *to which* L_Y *extends amply and universally, as in (5) (even (5')).*

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Remarks

- In particular, for a proper log smooth (X, D) with a polarizable $\mathbb{Z}$-VHS, $L_X = \bigotimes_p \det F^p V$ is semiample.
- Y^{BB} is stratified by subvarieties with quasifinite period maps—the ones associated to the associated graded of the limit mixed Hodge structures.
- Lots of previous work of Green–Griffiths–Laza–Robles and Green–Griffiths–Robles, including some special cases. Green–Griffiths–Robles establish key ingredient of our proof.

Other semipositive line bundles

Let $(W, F^\bullet W)$ on X° be a polarizable $\mathbb{Z}$-VHS whose deepest $F^p W$ is a line bundle, called the *Hodge bundle* M_{X°. We say $(W, F^\bullet W)$ is a CY $\mathbb{Z}$-VHS.

Example. If $\pi : Z^\circ \to X^\circ$ a smooth projective family of Calabi–Yau m-folds, $W = R^m \pi_* \mathbb{Z}_{(Z^\circ)^{\mathrm{an}}}$. Then $M_{X^\circ} = \pi_* \omega_{Z^\circ / X^\circ}$.

Question. Is M_X semiample for abstract CY $\mathbb{Z}$-VHS?

Not always!

But sometimes! In fact, for *any* polarizable $\mathbb{Z}$-VHS $(V, F^\bullet V)$, we may form

$$\mathsf{Griff}(V) := \bigotimes_p \bigwedge^{\mathrm{rk}\, F^p} V$$

This is a polarizable $\mathbb{Z}$-VHS whose Hodge bundle is the Griffiths bundle of V.

Main theorem 2

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

Let (X, D) be a proper log smooth algebraic space and $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on X°. Assume the Hodge bundle M_X is **integrable** *and* **has torsion combinatorial monodromy***. Then M_X is semiample.*

Integrability. If period map of M_X is not generically immersive on some subvariety, then the period map of the $\mathbb{Q}$-closure is not generically immersive.

Automatic for the Griffiths bundle

Torsion combinatorial monodromy. If M_X is numerically trivial on a connected curve, it is torsion.

Theorem (Green–Griffiths–Robles)

The Griffiths bundle has torsion combinatorial monodromy.

b-semiampleness

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ the polarizable CY $\mathbb{Z}$-VHS on X° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

For (Z, Δ) an lc pair and $\pi : Z \to X$ a fibration with $K_Z + \Delta \sim_\pi 0$, then

$K_Z + \Delta \sim \pi^*(K_X + B_X + M_X)$ (Kodaira, Kawamata, Fujino, Mori, Kollár,...)

Corollary (b-semiampleness conjecture of Prokhorov–Shokurov)

M_X is b-semiample.

Partial past results of Ambro, Lazić, Floris,...

Corollary

Moduli stacks of polarized klt CY pairs have canonical Baily–Borel compactifications, up to taking coarse space, reduction, normalization.

Thm 2, Step 1: make the topological space

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on X°. Assume the Hodge bundle M_X is **$(*)$ integrable** *and* **$(**)$ has torsion combinatorial monodromy**. *Then M_X is semiample.*

Let R be the equivalence relation on X of being connected by chains of M_X-degree zero curves.

Lemma

R is a proper algebraic equivalence relation. In particular, $Y = X/R$ exists as a reasonable topological space.

Moreover, natural stratification of (X, D) descends to Y—that is, Y has a stratification s.t. inverse images of strata Y_S are unions X_S of strata of X.

Key: By BBT, **each stratum** Y_S is algebraic and M_X descends amply. Here we use $(*) + (**)$.

Thm 2, Step 2: locally make **some** sections of M_X

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

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In a tubular neighborhood $T(S)$ of a union of strata X_S, there is a quotient $V \to V^{\min}(S)$ which extends over boundary

AND on the boundary, $V_S^{\min} := V^{\min}(S)|_{X_S}$ contains smallest subquotient of limit mixed Hodge structure V_S^{tr} containing M_X.

$$\begin{array}{ccc} \widetilde{T(S)}^{V^{\min}(S)} & \longrightarrow & \mathbb{P}V^{\min}_{S,x_s} \\ \cup & & \cup \\ \widetilde{X_S}^{V_S^{\min}} & \longrightarrow & \mathbb{P}V^{\mathrm{tr}}_{S,x_s} \end{array}$$

Key: AND $(*) + (**) \Rightarrow$ connected comp. of fibers on $\widetilde{X_S}^{V_S^{\min}}$ are **compact**.

Thm 2, Step 3: inductively glue and algebraize $Y = X/R$

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on X°. Assume the Hodge bundle M_X is $(*)$ **integrable** *and $(**)$* **has torsion combinatorial monodromy**. *Then M_X is semiample.*

Let $Y^{\leqslant i}$ be union of codimension $\leqslant i$ strata.

Problem. Local sections from Step 2 **DO NOT** give Y an analytic structure.

BUT, inductively assume global sections of M_X separate fibers over $Y^{<i}$.

These sections plus local sections from Step 2 **DO** imply $Y^{\leqslant i}$ has structure of **definable** analytic variety.

Definable GAGA (BBT) $\Rightarrow$ $Y^{\leqslant i}$ is **algebraic**.

(BBT) $\Rightarrow$ global sections of M_X separate fibers over $Y^{\leqslant i}$.QED

Thm 3: minimal lc centers

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ the polarizable CY $\mathbb{Z}$-VHS on X° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

For a lc fibration $\pi : (Z, \Delta) \to X$, a minimal lc center dominating X (="source") carries the $\mathbb{Q}$-closure V^{tr} of the Hodge bundle.

Key. Works well in the boundary too!

Thm 3: integrability

Essentially a result of Ambro.

Idea. For CY pairs, the period map of the Hodge bundle is immersive on the deformation space.

So if the Hodge bundle is trivial along a transcendental curve, the source must vary trivially, so V^{tr} is isotrivial.

Thm 3: torsion combinatorial monodromy

Problem. Source is not unique.

BUT Kollár's $\mathbb{P}^1$-linking $\Rightarrow$ V^{tr}s at node are glued via birational identification of sources $S_1 \simeq S_2$.

$$\mathrm{img}\left(\mathrm{Bir}(S_1, S_2) \to \mathrm{Hom}(H^0(\omega_{S_1}), H^0(\omega_{S_2})\right) < \infty$$

Thanks!