Baily–Borel compactifications of period images and the *b*-semiampleness conjecture

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The classical story: $\mathcal{A}_g = \operatorname{Sp}_{2\sigma}(\mathbb{Z}) \backslash \mathbb{H}_g$

- (1) (Satake '56) $A_{\sigma}^{\text{SBB}} = A_{\sigma} \sqcup A_{\sigma-1} \sqcup \cdots \sqcup A_1 \sqcup A_0$.
- (2) (Baily '58) $A_{\sigma}^{\text{SBB}} = \text{Proj}(\text{graded ring of automorphic forms}).$
- (3) modular interpretation.

$$Z^{\circ} \xrightarrow{} Z \longleftrightarrow Z_0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$C^{\circ} \xrightarrow{} C \qquad \Rightarrow \qquad 0$$

$$\xrightarrow{} Z^{\circ}/C^{\circ} \text{ a family of abelian varieties}$$

$$\bullet Z/C \text{ is the (semistable) Neron model}$$

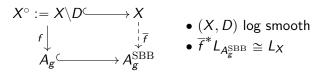
$$\bullet \text{ Identity comp. of } Z_0 \text{ is a semi-abel. variety}$$

$$\bullet \text{ Compact part is limiting point in boundary}$$

- (4) natural polarization. $\pi: Z \to \mathcal{A}_{\varphi}$ the universal family, $L=\det\pi_*\Omega_{Z/\mathcal{A}_\sigma}$ extends to an ample bundle $L_{A_\sigma^{\mathrm{SBB}}}$ (up to a power).

The classical story: $\mathcal{A}_g = \operatorname{Sp}_{2\sigma}(\mathbb{Z}) \backslash \mathbb{H}_g$

(5) universality. (Borel '72)



(5') Even have (5) in the analytic category, i.e. $(X, D) = (\Delta^k, \text{coordinate hyperplanes})$

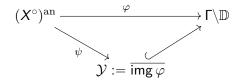
(Satake '60, Baily-Borel '66) Generalized to arbitrary arithmetic locally symmetric varieties.

Period maps

Let (X, D) log smooth proper

 $\pi:Z^{\circ} \to X^{\circ}$ be a smooth projective family,

 $V=R^k\pi_*\mathbb{Z}_{(Z^\circ)^{\mathrm{an}}}$ equipped with its polarizable \mathbb{Z} -VHS (filtration $F^\bullet V$ on $\mathcal{O}_{X^{\mathrm{an}}}\otimes_{\mathbb{C}_{X^{\mathrm{an}}}}V$).



Question (Griffiths '70).

- (A) Is \mathcal{Y} algebraic?
- (B) Is Griffiths bundle $L_{\mathcal{Y}} := \bigotimes_{p} \det F^{p}V$ algebraic? Ample?
- (C) Is there a \mathcal{Y}^{BB} ?

Main theorem 1

Question (Griffiths '70).

- (A) Is \mathcal{Y} algebraic?
- (B) Is $L_{\mathcal{Y}} := \bigotimes_{p} \det F^{p}V$ algebraic? Ample?

Theorem (B-Brunebarbe-Tsimerman '23)

$$\left((X^\circ)^{\mathrm{an}} \xrightarrow{\psi} \mathcal{Y}\right) = \left(X^\circ \xrightarrow{f} Y\right)^{\mathrm{an}} \text{ and } L_{\mathcal{Y}} = (L_Y)^{\mathrm{an}} \text{ all algebraic, } L_Y \text{ ample.}$$

(C) Is there a \mathcal{Y}^{BB} ?

$\overline{\mathsf{Theorem}\;1}\;(\mathsf{B}\mathsf{-Filipazzi-Mauri-Tsimerman})$

- $B_Y := \bigoplus_k H^0_{mg}(Y, L^k_Y)$ is finitely generated.
- $Y^{\mathrm{BB}} := \operatorname{Proj} B_Y$ is projective compactification of Y to which L_Y extends amply and universally, as in (5) (even (5')).

Theorem 1 (B–Filipazzi–Mauri–Tsimerman)

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Remarks

- In particular, for a proper log smooth (X, D) with a polarizable \mathbb{Z} -VHS, $L_X = \bigotimes_p \det F^p V$ is semiample.
- $m{\cdot}$ Y^{BB} is stratified by subvarieties with quasifinite period maps—the ones associated to the associated graded of the limit mixed Hodge structures.
- Lots of previous work of Green-Griffiths-Laza-Robles and Green-Griffiths-Robles, including some special cases.
 Green-Griffiths-Robles establish key ingredient of our proof.

Other semipositive line bundles

Let $(W, F^{\bullet}W)$ on X° be a polarizable \mathbb{Z} -VHS whose deepest $F^{\rho}W$ is a line bundle, called the *Hodge bundle* $M_{X^{\circ}}$. We say $(W, F^{\bullet}W)$ is a CY \mathbb{Z} -VHS.

Example. If $\pi: Z^{\circ} \to X^{\circ}$ a smooth projective family of Calabi–Yau m-folds, $W = R^m \pi_* \mathbb{Z}_{(Z^{\circ})^{\mathrm{an}}}$. Then $M_{X^{\circ}} = \pi_* \omega_{Z^{\circ}/X^{\circ}}$.

Question. Is M_X semiample for abstract CY \mathbb{Z} -VHS?

Not always!

But sometimes! In fact, for any polarizable \mathbb{Z} -VHS $(V, F^{\bullet}V)$, we may form

$$Griff(V) := \bigotimes_{p} \bigwedge^{rk F^{p}} V$$

This is a polarizable \mathbb{Z} -VHS whose Hodge bundle is the Griffiths bundle of V.

Main theorem 2

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

Let (X,D) be a proper log smooth algebraic space and $(V,F^{\bullet}V)$ a polarizable CY \mathbb{Z} -VHS on X° . Assume the Hodge bundle M_X is **integrable** and **has torsion combinatorial monodromy**. Then M_X is semiample.

Integrability. If period map of M_X is not generically immersive on some subvariety, then the period map of the \mathbb{Q} -closure is not generically immersive.

Automatic for the Griffiths bundle

Torsion combinatorial monodromy. If M_X is numerically trivial on a connected curve, it is torsion.

Theorem (Green–Griffiths–Robles)

The Griffiths bundle has torsion combinatorial monodromy.

b-semiampleness

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(X,D) proper log smooth, $(V,F^{\bullet}V)$ the polarizable CY \mathbb{Z} -VHS on X° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

For (Z,Δ) an lc pair and $\pi:Z\to X$ a fibration with $\mathcal{K}_Z+\Delta\sim_\pi 0$, then

 $K_Z + \Delta \sim \pi^*(K_X + B_X + M_X)$ (Kodaira, Kawamata, Fujino, Mori, Kollár,...)

Corollary (b-semiampleness conjecture of Prokhorov–Shokurov)

 M_X is b-semiample.

Partial past results of Ambro, Lazić, Floris,...

Corollary

Moduli stacks of polarized klt CY pairs have canonical Baily–Borel compactifications, up to taking coarse space, reduction, normalization.

Thm 2, Step 1: make the topological space

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

(X,D) proper log smooth, $(V,F^{\bullet}V)$ a polarizable $CY\mathbb{Z}\text{-}VHS$ on X° . Assume the Hodge bundle M_X is (*) integrable and (**) has torsion combinatorial monodromy. Then M_X is semiample.

Let R be the equivalence relation on X of being connected by chains of M_X -degree zero curves.

Lemma

R is a proper algebraic equivalence relation. In particular, Y = X/R exists as a reasonable topological space.

Moreover, natural stratification of (X, D) descends to Y—that is, Y has a stratification s.t. inverse images of strata Y_S are unions X_S of strata of X.

Key: By BBT, **each stratum** Y_S is algebraic and M_X descends amply. Here we use (*) + (**).

Thm 2, Step 2: locally make **some** sections of M_X

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

(X,D) proper log smooth, $(V,F^{\bullet}V)$ a polarizable CY \mathbb{Z} -VHS on X° . Assume the Hodge bundle M_X is (*) integrable and (**) has torsion combinatorial monodromy. Then M_X is semiample.

In a tubular neighborhood T(S) of a union of strata X_S , there is a quotient $V \to V^{\min}(S)$ which extends over boundary

AND on the boundary, $V_S^{\min} := V^{\min}(S)|_{X_S}$ contains smallest subquotient of limit mixed Hodge structure V_S^{tr} containing M_X .

$$\widetilde{T(S)}^{V^{\min}(S)} \longrightarrow \mathbb{P}V^{\min}_{S,x_s}$$
 $\widetilde{X_S}^{V_S^{\min}} \longrightarrow \mathbb{P}V^{\operatorname{tr}}_{S,x_s}$

Key: AND $(*) + (**) \Rightarrow$ connected comp. of fibers on $\widetilde{X_S}^{V_S^{\min}}$ are **compact**.

Thm 2, Step 3: inductively glue and algebraize Y = X/R

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

(X,D) proper log smooth, $(V,F^{\bullet}V)$ a polarizable CY \mathbb{Z} -VHS on X° . Assume the Hodge bundle M_X is (*) integrable and (**) has torsion combinatorial monodromy. Then M_X is semiample.

Let $Y^{\leqslant i}$ be union of codimension $\leqslant i$ strata.

Problem. Local sections from Step 2 **DO NOT** give Y an analytic structure.

BUT, inductively assume global sections of M_X separate fibers over $Y^{< i}$.

These sections plus local sections from Step 2 **DO** imply $Y^{\leq i}$ has structure of **definable** analytic variety.

Definable GAGA (BBT) $\Rightarrow Y^{\leqslant i}$ is algebraic.

(BBT) \Rightarrow global sections of M_X separate fibers over $Y^{\leqslant i}$.QED

Thm 3: minimal lc centers

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(X,D) proper log smooth, $(V,F^{\bullet}V)$ the polarizable CY \mathbb{Z} -VHS on X° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

For a lc fibration $\pi:(Z,\Delta)\to X$, a minimal lc center dominating X (="source") carries the \mathbb{Q} -closure V^{tr} of the Hodge bundle.

Key. Works well in the boundary too!

Thm 3: integrability

Essentially a result of Ambro.

Idea. For CY pairs, the period map of the Hodge bundle is immersive on the deformation space.

So if the Hodge bundle is trivial along a transcendental curve, the source must vary trivially, so $V^{\rm tr}$ is isotrivial.

Thm 3: torsion combinatorial monodromy

Problem. Source is not unique.

BUT Kollár's \mathbb{P}^1 -linking $\Rightarrow V^{\text{tr}}$ s at node are glued via birational identification of sources $S_1 \simeq S_2$.

$$\operatorname{img}\left(\operatorname{Bir}(S_1,S_2) \to \operatorname{Hom}(H^0(\omega_{S_1}),H^0(\omega_{S_2})\right) < \infty$$

Thanks!