BAILY–BOREL COMPACTIFICATIONS OF PERIOD IMAGES AND THE B-SEMIAMPLENESS CONJECTURE

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Abstract. We address two questions related to the semiampleness of line bundles arising from Hodge theory. First, we prove there is a functorial compactification of the image of a period map of a polarizable integral pure variation of Hodge structures for which the Griffiths bundle extends amply. In particular the Griffiths bundle is semiample. We prove more generally that the Hodge bundle of a Calabi–Yau variation of Hodge structures is semiample subject to some extra conditions, and as our second result deduce the b-semiampleness conjecture of Prokhorov–Shokurov. The semiampleness results (and the construction of the Baily–Borel compactifications) crucially use o-minimal GAGA, and the deduction of the b-semiampleness conjecture uses work of Ambro and results of Kollár on the geometry of minimal lc centers to verify the extra conditions.

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1. Introduction

1.1. **Baily–Borel compactifications of images of period map.** Let (X, D) be a log-smooth algebraic space, such that $X\backslash D$ supports a polarizable integral pure variation of Hodge structures $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$. Letting $\mathbb{D}$ be the associated period domain and Γ an arithmetic group containing the monodromy of $V_{\mathbb{Z}}$, we obtain a *period map* $\phi : (X\backslash D)^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$.

In general, the space $\Gamma\backslash\mathbb{D}$ cannot be equipped with an algebraic structure [GRT14]. Nonetheless, Griffiths conjectured [Gri70b, p.259] that the closure of the image of a period map is naturally a quasiprojective variety, which was proven by Griffiths assuming the image is proper, Sommese [Som75] in the case of isolated singularities, and in general in [BBT23a, Theorem 1.1]. We call $\overline{\phi((X\backslash D)^{\mathrm{an}})}$ a *period image*. One may therefore intrinsically define period images as closed Griffiths transverse algebraic subvarieties of $\Gamma\backslash\mathbb{D}$.

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In the classical case of Shimura varieties such as the moduli space of principally polarized abelian varieties $\mathcal{A}_g$, the space $\Gamma\backslash\mathbb{D}$ itself has an algebraic structure, and is therefore a universal period image. The celebrated work of Baily–Borel [BB66] provides a canonical projective compactification. It is therefore natural to ask whether there is an analogue of the Baily–Borel compactification for arbitrary period images.

Our first main result is to construct such a compactification (see Theorem 5.2 for the precise statement). As in the classical case, it satisfies the following extension property:

Theorem 1.1. *Let Y be a period image. Then there exists a functorial projective compactification Y^{BB} such that for any log smooth algebraic space (Z, D_Z), any morphism $Z\backslash D_Z \to Y$ for which the resulting morphism $(Z\backslash D_Z)^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ is locally liftable*[1] *extends to a morphism $Z \to Y^{\mathrm{BB}}$.*

It is easy to see this property uniquely determines Y^{BB} up to normalization. In fact, a stronger extension property holds with respect to analytic maps from punctured polydisks—see Theorem 1.9 below—which was proven in the classical case by Borel [Bor72].

1.1.1. *Griffiths bundle.* Returning to the variation on $X\backslash D$, there is a natural line bundle which descends to a polarization on Y and clarifies the statement of Theorem 1.1. It is provided by the Griffiths bundle:

$$L_{X\backslash D} = \bigotimes_p \bigwedge^{\mathrm{rk}\, F^p V_{\mathcal{O}}} F^p V_{\mathcal{O}}.$$

Importantly, this can be realized as the deepest piece of the Hodge filtration of the variation $\mathrm{Griff}(V) := \bigotimes_p \bigwedge^{\mathrm{rk}\, F^p V_{\mathcal{O}}} V$.

Each power $L^n_{X\backslash D}$ of the Griffiths bundle has a natural extension via the nilpotent orbit theorem of Schmid [Sch73] to a line bundle $(L^n_{X\backslash D})_X$ on all of X which is nef (and compatible with tensor power) if the local monodromy is unipotent and big if the period map is in addition generically immersive. It is essentially conjectured in [GGLR17] that L_X is semiample if the local monodromy is unipotent. We prove this conjecture, and use it to give an intrinsic characterization of Y^{BB} as follows.

It is easy to show [BBT23a, Lem 6.12] that a power of $L^m_{X\backslash D}$ descends to a line bundle $L_Y^{(m)}$. We define a section of $L_Y^{(mk)}$ to have *moderate growth* if its pullback to $X\backslash D$ for some (hence any) (X, D) extends to a section of L_X^{mk}. Finally, we define the ring of moderate growth sections $B_Y := \bigoplus_k H^0_{\mathrm{mg}}(Y, L_Y^{(mk)})$. We prove in Theorem 5.2 the following more precise statement:

Theorem 1.2 (Theorem 5.2)**.** *Let Y be a period image. Then*

(1) The ring B_Y is finitely generated, $Y^{\mathrm{BB}} := \mathrm{Proj}\, B_Y$ is a projective compactification of Y such that Theorem 1.1 holds, and the ample bundle $\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ (for sufficiently divisible n) naturally restricts to $L_Y^{(n)}$.

(2) Under the extension $Z \to Y^{\mathrm{BB}}$ from Theorem 1.1, $\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ (for sufficiently divisible n) pulls back to $(L^n_{Z\backslash D_Z})_Z$.

Corollary 1.3. *Let (X, D) be a log smooth algebraic space and V a polarizable integral pure variation of Hodge structures on $X\backslash D$ with unipotent local monodromy. Then the Griffiths bundle L_X of V is semiample.*

As in the classical case, it follows from the construction that Y^{BB} is stratified by locally closed subvarieties equipped with polarizable variations of Hodge structures with quasifinite period maps.[2]

[1] The local liftability is equivalent to the condition that $(Z\backslash D_Z)^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ be the period map of a variation. In fact, by the definability of period maps [BKT20, Theorem 1.3] and the definable Chow theorem of Peterzil–Starchenko [PS03, Corollary 4.5], such a morphism is equivalent to a variation on $Z\backslash D_Z$ whose period map factors through Y^{an}. If Γ is torsion-free the local liftability condition is automatic.

[2] This stratification and the associated variations are more complicated than those described in [GGLR17]. The stratification of Y^{BB} is not clearly canonical, and the variations mentioned above which descend from boundary strata in X will only be

1.1.2. *Past work.* Satake [Sat56] first constructed compactifications of Siegel modular varieties as ringed spaces, and suggested that they should in fact be analytic spaces. This was confirmed by Baily [Bai58], who moreover proved the compactifications were projective varieties, given as Proj of a finitely generated ring of automorphic forms. Baily–Borel [BB66] then constructed the analogous projective compactification of any Shimura variety, building on further work of Satake [Sat60b, Sat60a].

Shortly thereafter, Griffiths [Gri70b, §9] realized a full compactification of $\Gamma\backslash\mathbb{D}$ in the non-classical case is too much to hope for, and conjectured the existence of a *partial* compactification of $\Gamma\backslash\mathbb{D}$ with an extension property with respect to Griffiths transverse morphisms from log smooth sources as in Theorem 1.1. This idea provided motivation for the development of the theory of degenerations of Hodge structures (e.g. [Sch73, Ste75, Kas85, CKS86]) filling out a conjectural picture originally due to Deligne. Attempts have been made to construct such a partial compactification in some special cases, for example in the weight two case by Cattani–Kaplan [CK77], where the extension property was demonstrated for one-parameter period maps. A different perspective was proposed by Kato–Usui [KU09], who described a conjectural partial compactification of $\Gamma\backslash\mathbb{D}$ in the category of logarithmic manifolds, with the property that the closure of the image of any period map would be a proper algebraic space [Usu06] more closely analogous to the toroidal compactifications of [AMRT10]. This line of inquiry has been taken up recently by Deng [Den21, Den25] and Deng–Robles [DR23].

Griffiths [Gri70b, (10.7) Remark] also explicitly suggested an alternative generalization of the work of Baily–Borel (focusing on its relation to automorphic forms) would be to show that the ring of moderate growth sections of the Griffiths bundle on Y is finitely generated, and that its Proj provides a compactification of the period image. Theorem 1.2 confirms this expectation. The closely-related question of the semi-positivity of vector bundles of Hodge-theoretic origin has been considered by various authors starting with Griffiths [Gri70a] and continuing with, for example, Fujita, Zucker, and Kawamata [Fuj78, Zuc82, Kaw83]) (see also the references below regarding the canonical bundle formula).

The question of the semiampleness of the Griffiths bundle has been revived recently by Green, Griffiths, Laza, and Robles in [GGLR17], where Baily–Borel type compactifications and their the connection to other compactifications arising from moduli theory are discussed. Their work, together with subsequent work of Green–Griffiths–Robles [GGR25], has had an important influence on this one. Most notably, Green–Griffiths–Robles establish the torsion combinatorial monodromy of the Griffiths bundle in [GGR25, Thm 5.21] (see Theorem 2.22), which is a crucial ingredient. Implicit in their work is also the idea that the Griffiths bundle L_X is flat (with no residues) with respect to the natural logarithmic connection of the Deligne extension along any subvariety Z of X on which L_X is numerically trivial (see Remark 2.15). For us, this is ultimately upgraded to the existence of the local period maps described in (3) of the proof outline below. Finally, the $\dim X = 2$ case of Corollary 1.3 is proven in [GGLR17] and [GGR25]; the one-dimensional cases of Theorem 1.1 and Theorem 1.2 are easy.

1.1.3. *Applications.* As a corollary, we obtain a Baily–Borel compactification of any moduli space of polarized varieties with a local Torelli theorem.

Corollary 1.4 (of Theorem 5.2)**.** *Let $\mathcal{Y}$ be a reduced separated Deligne–Mumford stack equipped with a quasifinite period map. Then the coarse space Y has a compactification Y^{BB} for which some power $L_Y^{(n)}$ of the Griffiths bundle extends to an ample bundle $\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ and such that for any morphism $g : Z\backslash D_Z \to \mathcal{Y}$ for a log smooth algebraic space (Z, D_Z), the map on coarse spaces extends to $\overline{g} : Z \to Y^{\mathrm{BB}}$ with the property that $\overline{g}^*\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ pulls back to $(L^n_{Z\backslash D_Z})_Z$.*

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pieces of a certain tensor operation applied to the limit mixed Hodge structure. This is ultimately due to the lack of a canonical graded polarization on the limit mixed Hodge structure.

Corollary 1.4 for instance applies to any moduli stack of polarized varieties with an infinitesimal Torelli theorem, such as Calabi–Yau manifolds and most hypersurfaces (see [BBT23a]).

1.2. **b-semiampleness conjecture.** Let (Y, Δ) be a pair with log canonical singularities and $f\colon Y \to X$ a projective morphism with connected fibers to a normal variety such that $K_Y + \Delta \sim_{\mathbb{Q}} f^*L$ for a $\mathbb{Q}$-Cartier divisor L on X. Such a morphism is an example of an *lc-trivial fibration* (Definition 6.10); the condition essentially means the fibers are log Calabi–Yau pairs.

The *canonical bundle formula*, proven in increasing generality in [Kod66, Kod68, Fuj86, Kaw98, FM00, Amb04, Amb05, FG14b], implies that for an lc-trivial fibration, we can write

$$K_Y + \Delta \sim_{\mathbb{Q}} f^*(K_X + B_X + \mathbf{M}_X)$$

where:

- B_X is the *boundary divisor*, which is a $\mathbb{Q}$-divisor encoding the singularities of the degenerate fibers of f in codimension 1; and
- $\mathbf{M}_X$ is the *moduli part*, which is a $\mathbb{Q}$-b-divisor which is b-nef and encodes the variation of the generic fiber of the family.

Recall that a b-divisor is essentially an assignment of a divisor to any sufficiently high birational model of X. In particular, there is a modification $\pi\colon X' \to X$ and an lc-trivial fibration model $f'\colon Y' \to X'$ of the base-change of f for which $\mathbf{M}_{X'}$ is a nef $\mathbb{Q}$-divisor such that for any further modification $\pi'\colon X'' \to X'$ the moduli part pulls back, $\mathbf{M}_{X''} = \pi'^*\mathbf{M}_{X'}$. Such a model is called an Ambro model.

Our second main result is to prove the b-semiampleness conjecture by Prokhorov and Shokurov [PS09, Conj. 7.13.1]:

Theorem 1.5 (b-semiampleness)**.** *Let $f\colon (Y, \Delta) \to X$ be an lc-trivial fibration from a pair (Y, Δ) such that Δ is effective over the generic point of X. Then the moduli part $\mathbf{M}$ is b-semiample.*

Equivalently, $\mathbf{M}_{X'}$ is semiample for any Ambro model $f'\colon Y' \to X'$ of f. Note that the algebricity of Y and X can be dropped. Indeed, in Theorem 7.3, we show the b-semiampleness of the moduli part for projective morphisms of complex analytic spaces. Some immediate applications of the b-semiampleness conjecture are discussed in §7.2.

On a suitably chosen alteration of X, the moduli part coincides with the Schmid extension of the lowest Hodge filtration piece of the variation of Hodge structures on middle cohomology of the generic part of the family; see §6.3.[3] Thus, after some reductions, Theorem 1.5 is deduced from the following purely Hodge-theoretic result. By a CY variation of Hodge structures, we mean a variation of Hodge structures whose deepest nonzero Hodge filtration piece has rank one. We refer to this deepest piece (or its Schmid extension) as the *Hodge bundle*.

Theorem 1.6 (Theorem 4.1)**.** *Let (X, D) be a proper log smooth algebraic space, V a polarizable integral pure CY variation of Hodge structures on $X \backslash D$ with unipotent local monodromy, and M_X the Hodge bundle on X. If M_X is integrable with torsion combinatorial monodromy, then it is semiample.*

The integrability condition (see Definition 2.17) means that for any subvariety $Z \subset X$ for which the Hodge bundle is not big, there is some piece (defined over $\mathbb{Q}$) of the limit mixed Hodge structure variation on Z which contains the Hodge bundle and whose period map is not generically finite. The torsion combinatorial monodromy condition (see Definition 2.21) means that for any nodal curve $g\colon C \to X$ for which g^*M_X is numerically trivial (hence torsion on each component), it is in fact torsion. Both conditions are needed in the statement of Theorem 1.6—see Example 4.6 and Example 4.7—and therefore the proof that the conditions are satisfied in the case of Theorem 1.5 relies on the geometry of log Calabi–Yau pairs.

[3] A more algebraic characterization of the moduli part can be given as canonical moduli part of adjunction; see [Sho13, p.4].

1.2.1. *Past work.* Kodaira first studied the canonical bundle formula in the context of the classification of surfaces. In particular, the first instance of the formula is Kodaira's formula for the canonical bundle of a smooth minimal elliptic surface $S \to C$ [Kod66, Kod68]. Famously, in this case $\mathbf{M}_C = \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1)$, where j denotes the j-map $j \colon C \to \mathbb{P}^1$. Later, Fujita extended the formula to smooth varieties admitting an elliptic fibration [Fuj86]. In the case of elliptic fibrations over higher-dimensional bases, the j-map is not necessarily a morphism; thus, since Fujita's work, it became clear that to ensure positivity properties of the Hodge bundle, one must pass to a suitable higher birational model of the base.

In [Mor87, Rmk. 5.15.9.(ii)], Mori suggested the connection between the Hodge bundle of a family of Calabi–Yau varieties whose moduli are parametrized by a quotient of a Hermitian symmetric space. Fujino explored this direction in [Fuj03] and showed that the moduli divisor of an abelian- or K3-fibration is b-semiample. Fujino's work has then been extended by Kim to the case of primitive symplectic varieties [Kim25].

So far, the progress on the b-semiampleness conjecture has been limited. Prokhorov and Shokurov settled the general case in relative dimension 1 by leveraging knowledge of $\mathcal{M}_{0,n}$ [PS09]. Then, the works [Fil20, ABB$^+$23] settled the conjecture in relative dimension 2, complementing Fujino's work.

Works of Fujino–Mori and Ambro explored more general lc-trivial fibrations and settled weaker positivity properties of the moduli divisor; see [FM00, Amb04, Amb05]. While these earlier results only hold for fibrations with generically klt singularities, Fujino and Gongyo extended these results to fibrations with lc singularities [FG14b]. The idea of this latter work is to relate the Hodge bundle of the original fibration to the Hodge bundle of the fibration induced by the source of lc singularities. This is a key idea in the proof of Theorem 1.5.

In recent years, there has also been progress in the study of lc-trivial fibrations when the general fiber is not normal; see [Fuj22, FFL22]. In our work, we circumvent this problem by reducing to the fibration corresponding to minimal lc centers, which are necessarily normal.

Another important idea in this work is the study of the Hodge bundle along the subvarieties where it fails to be big. This direction has been originally explored in [FL19, Flo23]. In particular, in [Flo23], the author pursues similar ideas as in this work in considering the gluing of various period morphisms along snc configurations of subvarieties.

The relation between Baily–Borel compactifications of period images and the b-semiampleness conjecture has been clear to the experts, including, e.g., Ambro, Birkar, Fujino, Kollár, Mori, Prokhorov, and Shokurov. In the last decade, these ideas have been popularized by Laza in particular.

1.2.2. *Applications.* We also prove the existence of a Baily–Borel compactification as in Theorem 1.2 for the Hodge bundle, but subject to a normality condition.

Theorem 1.7 (Theorem 5.3)**.** *Let $\mathcal{Y}$ be a reduced separated normal Deligne–Mumford stack with a polarizable integral pure CY variation V. Assume that the Hodge bundle $M_{\mathcal{Y}}$ is strictly nef, integrable, and has torsion combinatorial monodromy. Let Y be the coarse space of $\mathcal{Y}$ and $M_Y^{(m)}$ the descent of some power of $M_{\mathcal{Y}}$. Then*

(1) The ring $C_Y \coloneqq \bigoplus_k H^0_{\mathrm{mg}}(Y, M_Y^{(mk)})$ is finitely generated, and $Y^{\mathrm{BBH}} \coloneqq \operatorname{Proj} C_Y$ is a normal projective compactification of Y for which the ample bundle $\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ (for sufficiently divisible n) restricts to $M_Y^{(n)}$.

(2) For a log smooth algebraic space (Z, D_Z) and any morphism $g : Z \setminus D_Z \to \mathcal{Y}$, the morphism on coarse spaces extends to $\overline{g} : Z \to Y^{\mathrm{BBH}}$ and $\overline{g}^* \mathcal{O}_{Y^{\mathrm{BB}}}(n)$ is identified with the Schmid extension $(M^n_{Z \setminus D_Z})_Z$. Moreover, Y^{BBH} is the unique normal compactification of Y for which some power of the Hodge bundle $M_Y^{(n)}$ extends to an ample line bundle and satisfies this property.

Experts, including Kollár and Shokurov, conjectured that the moduli part of an lc-trivial fibration should be the pullback of an ample $\mathbb{Q}$-divisor along a rational map to a compactified moduli space of the general fibers, as in the case of elliptic fibrations; see, e.g. [Kol07, §8.3.8] and [Sho13]. The compactification Y^{BBH} accomplishes it from a Hodge-theoretic viewpoint.

As a corollary, we prove the existence of a Hodge-theoretic compactification of the moduli space of smooth Calabi–Yau varieties on which the Hodge line bundle extends amply, which was conjectured by several authors, including, e.g., Odaka, Laza and Shokurov.

Corollary 1.8 (see Corollary 7.8)**.** *Let $\mathcal{Y}$ be a moduli stack of polarized smooth Calabi–Yau varieties. Then the coarse space Y has a unique normal compactification Y^{BBH} for which some power $M_Y^{(n)}$ of the Hodge bundle of the variation of Hodge structures on middle cohomology extends to an ample bundle $\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ and such that for any family $g : Z\backslash D_Z \to \mathcal{Y}$ for a log smooth algebraic space (Z, D_Z), the map on coarse spaces extends to $\overline{g} : Z \to Y^{\mathrm{BBH}}$ with the property that $\overline{g}^*\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ pulls back to $(M^n_{Z\backslash D_Z})_Z$.*

As in Theorem 1.2, the compactifications from Theorem 1.7 and Corollary 1.8 are stratified by varieties which are naturally equipped with CY variations with quasifinite period maps, so their codimension can be estimated from the numerics of the limit mixed Hodge structure—see Section 7.3.1. In fact, along codimension one strata, these variations are the transcendental part of the middle cohomology of the minimal lc center, or *source*; in general, they are obtained by iterating this procedure. In [Sho13, p.5], Shokurov predicted that the image of this compactified moduli map can be described in terms of the equivalence relation determined by having crepant birational sources; this seems plausible for Y^{BBH}, albeit only up to finite ambiguity, but we do not pursue this here. More on the Hodge theory of sources will appear in [Laz25]. The smoothness assumption in Corollary 1.8 may be dropped at the expense of taking the normalization of the coarse space Y—see Corollary 7.8. It would be interesting to compare the compactification Y^{BBH} with the conjectural compactification in [Oda22, Conj. B.1].

Note that in the classical case where Y is a Shimura variety, we may form both Y^{BB} and Y^{BBH}, and there is no difference between them. Indeed, for any level ≤ 2 polarizable variation of Hodge structures V with Hodge bundle $F^mV_{\mathcal{O}}$, the Griffiths bundle of V is a power of the Hodge bundle of $\bigwedge^{\mathrm{rk}\, F^mV_{\mathcal{O}}} V$, since $F^mV_{\mathcal{O}} \cong (\mathrm{gr}_F^{m-2} V_{\mathcal{O}})^\vee$ implies $\det F^mV_{\mathcal{O}} \cong \det F^{m-1}V_{\mathcal{O}}$ as $F^{m-2}V_{\mathcal{O}} = V_{\mathcal{O}}$ has trivial determinant. This also applies to the variation on middle cohomology for a family of surfaces. In general, for instance, for many moduli spaces of higher-dimensional Calabi–Yau varieties as in Corollary 7.8, they can be different—see Section 7.3.2. There is however always a morphism $Y^{\mathrm{BB}} \to Y^{\mathrm{BBH}}$.

We deduce a version of the Borel extension theorem for both Y^{BB} and Y^{BBH}:

Theorem 1.9 (Theorem 5.5)**.** *Let Y be as in Theorem 1.2 (resp. Theorem 1.7), and Y^{BB} (resp. Y^{BBH}) its Baily–Borel compactification with respect to the Griffiths (resp. Hodge) bundle. Then any analytic morphism $(\Delta^*)^k \to Y^{\mathrm{an}}$ for which $(\Delta^*)^k \to \Gamma\backslash\mathbb{D}$ is locally liftable extends to a morphism $\Delta^k \to Y^{\mathrm{BB,an}}$ (resp. $\Delta^k \to Y^{\mathrm{BBH,an}}$).*

A version of Borel extension showing that any holomorphic map $(\Delta^*)^k \to Y^{\mathrm{an}}$ to a period image[4] satisfying the local liftability extends meromorphically with respect to any compactification was proven by Deng [Den23] and also follows directly from the definability of the period map [BKT20] (see the proof of Theorem 5.5).

Lastly, Theorem 1.5 has applications to foundational statements in the MMP. Among others, it allows one to descend lc singularities along lc-trivial fibrations and to formulate adjunction and inversion thereof for arbitrary lc centers. For the precise statements, we refer the reader to §7.2.

[4]In fact, to a variety admitting a complex variation of Hodge structure with period map with discrete fibers.

1.3. **Proof outline.** In the classical case, Baily–Borel's proof [BB66] first builds the quotient space Y^{BB} set-theoretically as a union of Shimura varieties, then endows it with a sheaf of analytic functions provided by certain modular forms, and upgrades this to an algebraic structure using GAGA. In this more general setting, a quotient space Y^{BB} which is stratified by period images may be constructed[5], and each stratum is algebraic by [BBT23a]. On the one hand, there are algebraic sections of the Griffiths bundle on strata which separate points but are not "Hodge-theoretic" since they ultimately come from algebraic geometry rather than universally on $\Gamma\backslash\mathbb{D}$, so it is not clear how to lift them to a neighborhood of a stratum. On the other hand, analytic "Hodge-theoretic" sections may be constructed locally in Y^{BB} around any particular stratum, but their behavior off of that stratum is unclear. In particular, it is not clear there are enough such sections locally at a point of a stratum to separate points in a larger stratum specializing to it.

We blend these two perspectives and work inductively, adding strata one at a time, starting with the highest-dimensional stratum, showing (i) that the resulting space is a definable analytic space, and (ii) using definable GAGA [BBT23a] to algebraize it. Furthermore, we show this procedure can be carried out in general for a CY variation V given the hypotheses in Theorem 1.6, so the proofs of Theorem 1.2 and Theorem 1.6 are intertwined. Step (i) is achieved by the methods of [BBT23a, Theorem 5.4] combined with the construction of "Hodge-theoretic" sections which exist definably locally on the stratum using a certain part of the period map of the stratum which lifts to a neighborhood.

A more precise summary is as follows:

(1) We start with a proper log smooth algebraic space (X, D) equipped with a variation of Hodge structures V on $X\backslash D$. The algebraic space X comes equipped with a natural locally closed stratification $\{X_\Sigma\}$. We also form an auxiliary CY variation E on $X\backslash D$ with Hodge bundle M_X:

  (Thm 1.2) We take $E = \mathrm{Sym}^N \mathrm{Griff}(V)$ for an appropriate N, so M_X is the Nth power of the Griffiths bundle of V.

  (Thm 1.6) We take $E = V$.

(2) We define a proper algebraic equivalence relation R on X and show that we may modify (X, D) so as to make R as nice as possible. This allows us to construct a quotient space Z as a (definable) topological space with an induced stratification $\{Z_T\}$.

  (Thm 1.2) Let Y be the image of the period map associated to V. The closure of the equivalence relation on $X\backslash D$ defining the map $X\backslash D \to Y$, together with the relation of being connected by curves of degree 0 with respect to M_X, generates an equivalence relation R on X.

  (Thm 1.6) We take R to be the relation of being connected by curves of degree 0 with respect to M_X.

  Both of these relations must be proven to be algebraic.

(3) Each stratum X_Σ of X is naturally equipped with a variation of mixed Hodge structures E_Σ coming from the part of the limit mixed Hodge structure which is invariant under local monodromy. There is a smallest subquotient E^{tr}_Σ containing M_X, which is a pure variation, and a smallest quotient $E^{\min}_\Sigma$ containing M_X, which is a mixed variation. We naturally have $E^{\mathrm{tr}}_\Sigma \subset E^{\min}_\Sigma$, and the underlying local systems $E^{\mathrm{tr}}_{\Sigma,\mathbb{Q}}, E^{\min}_{\Sigma,\mathbb{Q}}$ extend to a tubular neighborhood $X_\Sigma \subset \mathfrak{T}_X(\Sigma) \subset X$.

  The full period map $\widetilde{X_\Sigma} \to \mathbb{D}_\Sigma$ of E^{tr}_Σ, where $\widetilde{X_\Sigma}$ is the minimal cover on which E^{tr}_Σ is trivialized, factors through some $\widetilde{Z_T}$ but does not lift to $\widetilde{\mathfrak{T}_X(\Sigma)}$. We show that the map $\widetilde{X_\Sigma} \to \mathbb{P}(E^{\min}_{\Sigma,\mathbb{C},x})$ obtained by only remembering $M_X \subset E^{\min}_\Sigma$ will: (i) have the same fibers along $\widetilde{X_\Sigma}$, and (ii) lift

[5] Although there are some subtleties—see Lemma 3.4, where we must use Lemma 2.20.

to $\widetilde{\mathfrak{T}_X(\Sigma)}$. This construction provides the definable analytic "Hodge-theoretic" sections described above definably locally on Z and crucially uses the fact that we are working with a CY variation, which is why even in the case of Theorem 1.2 it is important to consider the auxiliary CY variation E. It is also in this step that, in the case of Theorem 1.6, we use the integrability to ensure (i) and the torsion combinatorial monodromy condition to ensure that $\widetilde{X_\Sigma} \to \mathbb{P}(E^{\min}_{\Sigma,\mathbb{C},x})$ has compact fibers.

(4) Proceeding inductively, we algebraize larger and larger unions of strata. Suppose $U \subset Z$ is an open union of strata and $Z_T \subset U$ a stratum which is closed in U such that $U' \coloneqq U \backslash Z_T$ has been algebraized. To algebraize U, we first use [BBT23a, Theorem 5.4] to produce global sections of a power of M_X which separate fibers of $X \to Z$ over U'. Combining these with the sections of M_X coming from (3) which exist definably locally on Z_T and separate points on Z_T, we obtain a definable analytic projective embedding definably locally on U (using the compactness of the fibers of the map in (3)), and the definable analytic structures of the images glue to give a definable analytic structure on U. This then gives an algebraic structure on U by [BBT23a, Theorem 1.3] to which a power of M_X descends to a line bundle $M_U^{(m)}$ by definable GAGA which is moreover ample by [BBT23a, Theorem 5.4] again. By induction, the quotient map $X \to Z$ is algebraic and a power of M_X descends to an ample bundle $M_Z^{(m)}$ on Z.

(5) This completes the proof of Theorem 1.6, and Theorem 1.7 easily follows since in this case the algebraic structure on Z may be taken to be normal. In the case of Theorem 1.2, having equipped Z with *some* algebraic structure compactifying Y and such that (a power of) the Griffiths bundle L_Y extends to an ample bundle which pulls back to L_X, it is now an easy noetherianity argument to show that B_Y is finitely generated and its Proj gives a compactification satisfying the properties in Theorem 1.2.

Theorem 1.7 is easier than the corresponding parts of Theorem 1.2 because of the normality assumption. Indeed, in this case, the quotient map $X \to Z$ may be assumed to have connected fibers, so it admits a preferred algebraization for which the morphism is a fibration, and this algebraic structure is determined by the underlying topology. In reality, we first prove Theorem 1.7, deduce Theorem 1.2 for the normalization of Y from it, and then prove Theorem 1.2 for Y itself, as it simplifies the argument. The descent along the normalization is delicate and critically uses the fact that E comes from V via the $\mathrm{Griff}(-)$ construction.

Finally, the b-semiampleness conjecture (Theorem 1.5) follows from Theorem 1.6, provided we verify that a "geometric" Hodge bundle, i.e., coming from an lc-trivial fibration $f\colon (Y, \Delta) \to X$, is automatically integrable with torsion combinatorial monodromy. To this end, we explore the geometric significance of these two conditions.

(6) The key geometric input is the notion of *source* of an lc-trivial fibration, roughly the smallest stratum of (Y, Δ) dominating the base; see Definition 6.9 for the formal definition, and cf. also [Kol13, §4.5]. We study how the moduli part of $f\colon (Y, \Delta) \to X$ restricts to a prime divisor $D_X \subset X$. Up to an alteration of the base, we identify the restriction to D_X of the moduli part of f with the moduli part of the source $(S, \Delta_S) \to D_X$ of the restricted lc-trivial fibration $(Y, \Delta) \times_{D_X} X \to D_X$.
(7) The integrability of the Hodge bundle follows from the fact that the variation of the source is maximal if and only if its Hodge bundle is big; cf. [Amb05].
(8) The torsion combinatorial monodromy is a consequence of the isotriviality of lc-trivial fibrations with torsion moduli part (over a normal base) due to [Amb05], and the finiteness of b-representations, proved in increasing level of generality by [NU73, Fuj00, Gon13, HX16, FG14a]. The latter gives that the group $\mathrm{Bir}(F, \Delta_F)$ of crepant birational automorphisms of the general fiber (F, Δ_F) of $(S, \Delta_S) \to D_X$ acts finitely on $H^0(F, \omega_F^{[m]}(m\Delta_F))$ for $m \gg 1$. Now, to check torsion combinatorial monodromy,

consider a testing nodal curve $C \to X$ such that the moduli part vanishes on each irreducible component of C. Up to some technical reductions, the sources of the pullback family $Y_C \to C$ are isotrivial families along each irreducible of C, connected by crepant birational automorphisms of their fibers over the nodes of the C by Kollár's theory of $\mathbb{P}^1$-linking [Kol13, §4.4]. Hence, the monodromies of multiples of the Hodge bundle factor through the finite b-representations of $\mathrm{Bir}(F, \Delta_F)$, which entails the torsion of the combinatorial monodromy.

1.4. **Paper outline.** The paper is organized as follows.

(§2) We recall the variations of (monodromy invariant) limiting mixed Hodge structures one obtains from a variation V on each stratum of a log smooth space (X, D), and introduce their transcendental and CY-minimal quotient pieces. We define the integrability and torsion combinatorial monodromy conditions and explain why both conditions are satisfied in the case of the Griffiths bundle. The latter is a result of [GGR25].

(§3) We discuss the algebraicity of equivalence relations of Hodge-theoretic nature as in (2) of the proof outline. We also prove some lemmas regarding refinement of log smooth spaces with respect to these data. Finally, we construct the maps $\widetilde{X_\Sigma} \to \mathbb{P}(E^{\min}_{\Sigma,\mathbb{C},x})$ from (3) of the proof outline.

(§4) We prove the semiampleness statement in Theorem 1.2 (i.e. Corollary 1.3) and Theorem 1.6 as in (4) of the proof outline.

(§5) We deduce Theorem 1.7 and prove Theorem 1.2 as in (4) and (5) of the proof outline. We also prove the Borel extension result Theorem 1.9.

(§6) We recall the notion of moduli part and source of an lc-trivial fibration. We compare the variation of Hodge structures associated to them, and we make some preliminary reductions for the proof of Theorem 1.5.

(§7) Using the preparations from §6, we verify the integrability and torsion combinatorial monodromy conditions for the moduli part to deduce Theorem 1.5. Then we discuss applications of the b-semiampleness conjecture in birational geometry (§7.2) and for the moduli theory of log Calabi–Yau pairs (§7.3).

Notation.

- Throughout, analytic spaces, definable analytic spaces, and algebraic spaces are always taken over $\mathbb{C}$ and to be separated. Algebraic spaces are always of finite type over $\mathbb{C}$.
- By a log smooth algebraic space or snc pair (X, D) we mean the datum of a smooth algebraic space X together with a divisor $D \subset X$ with simple normal crossings.
- Throughout by a fibration we mean a proper morphism $f : X \to Y$ between normal spaces such that the pullback map $\mathcal{O}_Y \xrightarrow{\cong} f_*\mathcal{O}_X$ is an isomorphism.
- CY variations play the central role in the paper, so we refer to them by the letter V in §2-4,6 and typically use the letter L for the Hodge bundle. In the case of Theorem 1.2 where the relevant CY variation is obtained by the $\mathrm{Griff}(-)$ construction, we refer to the original variation as ${}_{\mathrm{orig}}V$, so $V = \mathrm{Griff}({}_{\mathrm{orig}}V)$. Most of §5 is devoted to the Griffiths case, so we once again reserve L for the Griffiths bundle and use M for the Hodge bundle.

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2. Hodge-theoretic preliminaries

2.1. **Definable analytic spaces.** We shall use the notion of definable analytic spaces from [BBT23a] freely. As in the majority of that paper, we shall only ever work with the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$. For the convenience of the reader, we reproduce here the three main results from [BBT23a] we will be using.

Theorem 2.1 (Definable GAGA, [BBT23a, Theorem 1.4])**.** *Let X be an algebraic space and X^{def} the associated definable analytic space. The definabilization functor $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{def}})$ is fully faithful, exact, and its essential image is closed under subobjects and quotients.*

Theorem 2.2 (Definable images, [BBT23a, Theorem 1.3])**.** *Let X be an algebraic space and $\phi : X^{\mathrm{def}} \to \mathcal{Z}$ a proper morphism of definable analytic spaces. Then there is a factorization*

$$\begin{tikzcd} X^{\mathrm{def}} \arrow[rr, "\phi"] \arrow[dr, "f^{\mathrm{def}}"'] & & \mathcal{Z} \\ & Y^{\mathrm{def}} \arrow[ur, "\iota"'] & \end{tikzcd}$$

where $f : X \to Y$ is a proper dominant[6] *morphism of algebraic spaces and $\iota : Y^{\mathrm{def}} \to \mathcal{Z}$ is a closed embedding of definable analytic spaces. Moreover, f is uniquely determined as a morphism with fixed source.*

Setup 2.3. Let L_Y be a line bundle on an algebraic space Y with the following property. For every reduced closed subspace $Z \hookrightarrow Y$ and any proper log smooth algebraic space (X, D) with a proper birational morphism $\pi : X \backslash D \to Z$, the pullback $L_{X\backslash D} \coloneqq \pi^* L_Z$ of the restriction L_Z extends to a nef and big line bundle L_X on X. Moreover, for any two such $(X, D), (X', D')$ and a morphism $g : (X', D') \to (X, D)$ with $\pi \circ g|_{X'\backslash D'} = \pi'$ we have $g^* L_X \cong L_{X'}$.

Definition 2.4. Assume Setup 2.3. Given a closed subscheme $Z \hookrightarrow Y$, we say a section s of L_Z^n *vanishes at the boundary* if for some (hence any) (X, D) as above the section s pulls backs and extends to a section of $L_X^n(-D)$. We let $H^0_{\mathrm{van}}(Z, L_Z^n) \subset \Gamma(Z, L_Z^n)$ denote the linear subspace of sections vanishing at the boundary, which is finite-dimensional as $H^0_{\mathrm{van}}(Z, L_Z^n)$ injects into $H^0(X, L_X^n)$.

Theorem 2.5 ([BBT23a, Theorem 5.4])**.** *Assume Setup 2.3. Then Y is a scheme and L_Y is an ample line bundle. Moreover, for every $n \gg 1$, the natural morphism $Y \to \mathbb{P}(H^0_{\mathrm{van}}(Y, L_Y^n)^\vee)$ is defined everywhere and is a locally closed embedding.*

Finally, the following notions will be useful; see [BBT24, §3] for more details.

Definition 2.6. For a definable analytic space $\mathcal{X}$ with a choice of basepoint $x \in \mathcal{X}$, a definable analytic space $\mathcal{P}$ admitting an action of $\pi_1(\mathcal{X}, x)$ by definable analytic automorphisms, and a covering space $\widetilde{\mathcal{X}} \to \mathcal{X}^{\mathrm{an}}$, we say a $\pi_1(\mathcal{X}, x)$-equivariant analytic morphism $\phi : \widetilde{\mathcal{X}} \to \mathcal{P}^{\mathrm{an}}$ is π_1-*definable analytic* if its restriction to any continuous lift of a definable open $\mathcal{U} \subset \mathcal{X}$ is definable analytic. A π_1-*definable analytic coherent sheaf* $\mathcal{E}$ on $\widetilde{X}$ is an analytic coherent sheaf which is equipped with the structure of a definable analytic coherent sheaf on any such lift $\mathcal{U}$, compatibly with respect to intersections and the $\pi_1(\mathcal{X}, x)$ action. It is clear that this is equivalent to a definable analytic coherent sheaf on $\mathcal{X}$.

Observe that if $\mathfrak{X}$ is a definable topological space and $\widetilde{\mathfrak{X}}$ a covering space, a $\pi_1(\mathfrak{X}, x)$-equivariant sheaf of $\mathbb{C}$-algebras $\mathcal{O}$ on $\widetilde{\mathfrak{X}}$ (with respect to covers pulled back from $\mathfrak{X}$), which is the structure sheaf of a definable analytic space on any continuous lift of a definable open subset of $\mathfrak{X}$, is equivalent to a definable analytic space structure on $\mathfrak{X}$.

[6]Here we mean "scheme-theoretically" dominant, that is, $X \to Y$ is surjective on points and $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is injective.

2.2. **Trivializing covers of local systems.** For a local system V on an analytic space X, we denote by $\widetilde{X}^V \to X$ the minimal covering space of X on which V is trivialized—that is, the cover corresponding to the quotient of $\pi_1(X,x)$ given by the monodromy representation $\pi_1(X,x) \to \mathrm{GL}(V_x)$ for some choice of basepoint x. If the analytic space is X^{an} for an algebraic space X, we denote the cover by $\widetilde{X}^V$ (as opposed to $\widetilde{X^{\mathrm{an}}}^V$).

2.3. **DR-neighborhoods.** Let X be a definable topological space. For any locally closed definable topological space Z, we say that a neighborhood $Z \subset \mathfrak{T}_X(Z) \subset X$ is a *DR-neighborhood* of Z if it has a strong deformation retraction onto Z.

Lemma 2.7. *Let X be a definable topological space, $i : Z \to X$ the inclusion of a locally closed definable subspace, $j : X\backslash Z \to X$ the inclusion of the complement, $\mathfrak{T}(Z) := \mathfrak{T}_X(Z)$ a DR-neighborhood of Z, and E a local system on $X\backslash Z$. For any subsheaf $G_Z \subset i^*j_*E$ which is a local system, there is a unique subsheaf $G(Z) \subset j_*E|_{\mathfrak{T}(Z)}$ which is a local system for which $i^*G(Z) = G_Z$.*

Proof. Without loss of generality, we may assume that both Z and $X\backslash Z$ are connected and nonempty, and $X = \mathfrak{T}(Z)$. Take $z \in Z$. For any $x \in X\backslash Z$, pick a path γ such that $\gamma(0) = z, \gamma(1) = x, \gamma((0,1]) \subset X\backslash Z$, and define $G^\gamma(Z)_x \subset E_z$ to be the space of elements which are the restriction of a section of $j_*E(U)$ on any neighborhood U containing γ.

We claim this is independent of γ. To see this, suppose γ' is another such path. Since X strongly deformation retracts onto Z, there is a loop γ_0 in Z based at z such that γ and $\gamma'' := \gamma\gamma_0$ are homotopic (as paths from z to x), and it suffices to replace γ' with γ''. Thus we may assume there is a homotopy H between γ and γ' such that $H\big((0,1] \times [0,1]\big) \subset X\backslash Z$. Then by continuing along H we see that $G^\gamma(Z)_x = G^{\gamma'}(Z)_x$. We thus obtained a well-defined local system $G(Z) \subset E$, and it is clear that $G(Z)$ extends to a subsheaf of j_*E as required. $\square$

If (X, D) is a log smooth algebraic space and $Z \subset X$ a connected component of an intersection of irreducible components of D, we may take $\mathfrak{T}_X(Z)$ to have the property that the retraction preserves D and that the resulting map $\mathfrak{T}_X(Z) \setminus D \to Z$ strongly deformation retracts onto a torus fibration.

2.4. **Griffiths and Hodge bundles.** Let (X, D) be a proper log smooth algebraic space. For a complex local system $V_{\mathbb{C}}$ on $(X\backslash D)^{\mathrm{an}}$ we denote by $V_{\mathcal{O}}$ the algebraic flat vector bundle on $X\backslash D$ (with regular singularities) corresponding to $V_{\mathbb{C}}$ via the Riemann–Hilbert correspondence, so $(V_{\mathcal{O}}, \nabla)^{\mathrm{an}} \cong (V_{\mathcal{O}^{\mathrm{an}}}, \nabla) := (\mathcal{O}_{(X\backslash D)^{\mathrm{an}}} \otimes V_{\mathbb{C}}, d\otimes 1)$. We denote by $(\mathcal{V}, \nabla)$ the Deligne extension of $(V_{\mathcal{O}}, \nabla)$, that is, the unique logarithmic flat vector bundle on (X, D) whose analytification extends the flat vector bundle $(V_{\mathcal{O}^{\mathrm{an}}}, \nabla)$ whose residues have eigenvalues with real part contained in $(-1, 0]$. If $j : X\backslash D \to X$ is the inclusion, we have a natural inclusion $j_*^{\mathrm{an}} V_{\mathbb{C}} \hookrightarrow \mathcal{V}^{\mathrm{an}}$.

Definition 2.8. Let (X, D) be a proper log smooth algebraic space. Let $(V_{\mathbb{C}}, F^\bullet V_{\mathcal{O}})$ be a graded polarizable admissible mixed complex variation of Hodge structures on $X\backslash D$ and $(\mathcal{V}, F^\bullet\mathcal{V})$ its Deligne/Schmid extension to X. We say $(V_{\mathbb{C}}, F^\bullet V_{\mathcal{O}})$ is a CY variation if the deepest nonzero part of the Hodge filtration $F^m V_{\mathcal{O}} \cong \mathrm{gr}_F^m V_{\mathcal{O}}$ has rank one, in which case we call $L = F^m\mathcal{V} \cong \mathrm{gr}_F^m \mathcal{V}$ the Hodge bundle.

We will need a version of the transcendental part of a CY variation.

Definition 2.9. Let (X, D) be a proper log smooth algebraic space and $V = (V_{\mathbb{Q}}, W_\bullet V_{\mathbb{Q}}, F^\bullet V_{\mathcal{O}})$ an admissible graded polarizable rational mixed variation of Hodge structures on $X\backslash D$. Let $F^m V_{\mathcal{O}} \neq 0$ be the smallest nonzero piece of the Hodge filtration and assume $\mathrm{rk}\, F^m V_{\mathcal{O}} = 1$.

(1) There is a unique minimal quotient $V \to V^{\min}$ in the category of rational mixed variations for which $\mathrm{gr}_F^m V_{\mathcal{O}}^{\min} \neq 0$, which factors through all other such quotients. We refer to $V^{\min}$ as the CY-minimal quotient.

(2) There is a unique minimal subvariation $U \subset V$ with $\mathrm{gr}_F^m U_{\mathcal{O}} \neq 0$ which we call the CY-minimal subobject. We don't introduce notation for the CY-minimal subobject.
(3) The CY-minimal subobject of the CY-minimal quotient of V is canonically identified with the CY-minimal quotient of the CY-minimal subobject of V; we call it the transcendental part V^{tr}. It is naturally identified with the smallest Hodge substructure $V^{\mathrm{tr}} \subset \mathrm{gr}_k^W V$ for which $\mathrm{gr}_F^m V^{\mathrm{tr}} \neq 0$, where k is the unique weight with $\mathrm{gr}_F^m \mathrm{gr}_k^W V \neq 0$.

Note that if V is a graded polarizable mixed Hodge structure with deepest nonzero Hodge filtration piece $F^m V_{\mathbb{C}}$ of rank 1, there is a unique k such that $\mathrm{gr}_F^m \mathrm{gr}_k^W V_{\mathbb{C}} \neq 0$. Then $V \to V^{\min}$ factors through the quotient $V/W_{k-1}V$, $U \subset W_k V$, and V^{tr} is both the lowest weight subobject of $V^{\min}$ and the highest weight quotient of U. Moreover, V^{tr} is simple, since $W_k V^{\min}$ is a pure polarizable Hodge structure.

For a polarizable rational pure variation V which is not necessarily CY with deepest nontrivial Hodge filtration piece $F^m V_{\mathcal{O}}$, we will sometimes refer to $V^{\mathrm{tr}} \subset V$ as the smallest rational subvariation for which $F^m V^{\mathrm{tr}} = F^m V$.

Lemma 2.10. *Let V be a graded-polarizable rational mixed Hodge structure such that the smallest nonzero piece $F^m V_{\mathbb{C}}$ of the Hodge filtration has $\mathrm{rk}\, F^m V_{\mathbb{C}} = 1$. Let $V \to V^{\min}$ be the CY-minimal quotient and $V^{\mathrm{tr}} \subset V^{\min}$ the transcendental part. Then*

$$\mathrm{Aut}(V^{\min}) \hookrightarrow \mathrm{Aut}(V^{\mathrm{tr}}) \hookrightarrow \mathrm{Aut}(\mathrm{gr}_F^m V_{\mathbb{C}}) = \mathrm{Aut}(\mathrm{gr}_F^m V_{\mathbb{C}}^{\min}) = \mathrm{Aut}(\mathrm{gr}_F^m V_{\mathbb{C}}^{\mathrm{tr}})$$

via the restriction maps, where the first two groups are automorphisms in the category of rational mixed Hodge structures and the last three are automorphisms in the category of vector spaces.

Proof. For any f in the kernel of either of the above two restriction maps, $\ker(f - \mathrm{id})$ is a Hodge substructure with $\mathrm{gr}_F^m \ker(f - \mathrm{id}) = 0$, which must therefore be trivial. □

In the general context of variations of Hodge structures, we will pass to a certain CY variation whose Hodge bundle detects variation in any filtration piece of the original variation.

Definition 2.11. Let (X, D) be a proper log smooth algebraic space. Let ${}_{\mathrm{orig}}V = ({}_{\mathrm{orig}}V_{\mathbb{Z}}, F^\bullet_{\mathrm{orig}} V_{\mathcal{O}})$ be a polarizable pure integral variation of Hodge structures on $X \backslash D$. We define

$$V = \bigotimes_p \bigwedge^{\mathrm{rk}\, F^p_{\mathrm{orig}} V_{\mathcal{O}}} {}_{\mathrm{orig}}V.$$

It is a polarizable integral pure CY variation of Hodge structures which has unipotent local monodromy if ${}_{\mathrm{orig}}V$ does. We refer to the Hodge bundle $L = \bigotimes_p \det F^p_{\mathrm{orig}} \mathcal{V}$ of V as the Griffiths bundle of ${}_{\mathrm{orig}}V$.

2.5. **Boundary variations.** Let (X, D) be a proper log smooth algebraic space. After a modification, we may assume the irreducible components of D are smooth, and that the intersection of any number of irreducible components of D is connected (though possibly empty). We call such a space a proper *strictly* log smooth algebraic space.

Note that the set of irreducible components of D is naturally identified with $\pi_0(D^{\mathrm{reg}})$ by taking closure. Moreover, by the above assumption, any component of the natural locally closed stratification of X induced by D can be uniquely characterized by which irreducible components of D they are contained in, or, equivalently, the dual complex of D is simplicial. Thus, we make the following definition: for any subset $\Sigma \subset \pi_0(D^{\mathrm{reg}})$, we take $X_\Sigma := \bigcap_{E \in \Sigma} \overline{E} \backslash \bigcup_{E \notin \Sigma} \overline{E}$.

Let $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ be a polarizable integral pure variation of Hodge structures of weight w on $X_\varnothing = X \backslash D$. We collect here the various variations one obtains on the strata by degenerating; see, for example, [PS08] for background.

2.5.1. *For each stratum X_Σ there is:*

- A DR-neighborhood $X_\Sigma \subset \mathfrak{T}_X(X_\Sigma) =: \mathfrak{T}(\Sigma)$ in the sense of Section 2.3. Set $\mathfrak{T}^*(\Sigma) := \mathfrak{T}(\Sigma)\backslash D$.
- For each $E \in \Sigma$, there is a globally defined nilpotent operator $N_E : V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)} \to V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}$ given by the logarithm of the unipotent part of the local monodromy around E. Indeed, there is a dense Zariski open set $U \subset X$ meeting X_Σ on which every divisor in Σ has a local defining equation. If $\mathfrak{T}(U \cap X_\Sigma)$ is a bundle over $U \cap X_\Sigma$ with fiber $(\Delta^*)^\Sigma$, then we have a commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \pi_1((\Delta^*)^\Sigma) \arrow[r] \arrow[d, equal] & \pi_1(U \cap \mathfrak{T}^*(\Sigma)) \arrow[r] \arrow[d, two heads] & \pi_1(U \cap \mathfrak{T}(\Sigma)) \arrow[r] \arrow[d, two heads] & 1 \\ & \pi_1((\Delta^*)^\Sigma) \arrow[r] & \pi_1(\mathfrak{T}^*(\Sigma)) \arrow[r] & \pi_1(\mathfrak{T}(\Sigma)) \arrow[r] & 1 \end{tikzcd} \tag{2.1}$$

where the top row is split by the defining equations. Hence, the meridian winding around E is central in $\pi_1(\mathfrak{T}^*(\Sigma))$, and so N_E intertwines the monodromy representation of $V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}$.
- Associated to the local monodromy logarithms $\{N_E\}_{E\in\Sigma}$ there is a weight filtration $W(\Sigma)_\bullet V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}$ on $V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}$ for which each N_E is degree -2. Each one has a natural saturated integral structure which we denote $W(\Sigma)_\bullet V_\mathbb{Z}|_{\mathfrak{T}^*(\Sigma)}$.
- The Hodge filtration $F^\bullet V_\mathcal{O}$ extends to a locally split filtration $F^\bullet\mathcal{V}$ of the Deligne extension $\mathcal{V}$.
- The restriction of Deligne extension $\mathcal{V}|_{\mathfrak{T}(\Sigma)}$ is naturally filtered by sub-logarithmic flat vector bundles $W(\Sigma)_\bullet\mathcal{V}|_{\mathfrak{T}(\Sigma)}$ which are the Deligne extensions of the local systems $W(\Sigma)_\bullet V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}$. There are natural flat morphisms $N_E : \mathcal{V}|_{\mathfrak{T}(\Sigma)} \to \mathcal{V}|_{\mathfrak{T}(\Sigma)}$ with degree -2 with respect to $W(\Sigma)_\bullet\mathcal{V}|_{\mathfrak{T}(\Sigma)}$.

2.5.2. *If we further suppose $V_\mathbb{Z}$ has unipotent local monodromy, then:*

- Each $\operatorname{gr}_k^{W(\Sigma)} V_\mathbb{Z}|_{\mathfrak{T}^*(\Sigma)}$ extends as a local system to $\mathfrak{T}(\Sigma)$. We somewhat abusively denote the extension by $\operatorname{gr}_k^{W(\Sigma)} V_\mathbb{Z}$.
- Define
$$V(\Sigma)_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)} := \operatorname{coker}\left(\oplus_{E\in\Sigma} N_E : \bigoplus_{E\in\Sigma} V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)} \to V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}\right)$$
which has a natural integral structure $V(\Sigma)_\mathbb{Z}|_{\mathfrak{T}^*(\Sigma)}$ coming from the image of $V_\mathbb{Z}|_{\mathfrak{T}^*(\Sigma)}$, and a natural filtration $W_\bullet V(\Sigma)_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}$ coming from the image of $W(\Sigma)_\bullet V_\mathbb{Q}|_{\mathfrak{T}^*(\Sigma)}$. Then $V(\Sigma)_\mathbb{Z}|_{\mathfrak{T}^*(\Sigma)}$ (together with its filtration) extends to $V(\Sigma)_\mathbb{Z}$ on $\mathfrak{T}(\Sigma)$.

  Observe that $\operatorname{gr}_k^W V(\Sigma)_\mathbb{Q}$ is naturally identified with the primitive part of $\operatorname{gr}_k^{W(\Sigma)} V_\mathbb{Q}$ with respect to the (simultaneous) hard Lefschetz decomposition with respect to the maps $N_E : \operatorname{gr}_k^{W(\Sigma)} V_\mathbb{Q} \to \operatorname{gr}_{k-2}^{W(\Sigma)} V(\Sigma)_\mathbb{Q}$ for $E \in \Sigma$.
- The graded pieces $\operatorname{gr}_k^{W(\Sigma)} \mathcal{V}|_{\mathfrak{T}(\Sigma)}$ of the Deligne extension have connection with no residue and are identified with the Deligne extensions of $\operatorname{gr}_k^{W(\Sigma)} V_\mathbb{Z}$. The same is true for
$$V(\Sigma)_\mathcal{O} := \operatorname{coker}\left(\oplus_{E\in\Sigma} N_E : \bigoplus_{E\in\Sigma} \mathcal{V}|_{\mathfrak{T}(\Sigma)} \to \mathcal{V}|_{\mathfrak{T}(\Sigma)}\right).$$
- $V_\Sigma := (V(\Sigma)_\mathbb{Z}|_{X_\Sigma}, W_\bullet V(\Sigma)_\mathbb{Q}|_{X_\Sigma}, F^\bullet V(\Sigma)_\mathcal{O}|_{X_\Sigma})$ is an admissible graded polarizable integral variation of mixed Hodge structures.

2.5.3. *If finally V is a CY variation with unipotent local monodromy and Hodge bundle $F^m V_\mathcal{O}$, then:*

- For each Σ there is a unique integer k_Σ such that $\operatorname{gr}_F^m \operatorname{gr}_{k_\Sigma}^W V_\Sigma \neq 0$. Then $\operatorname{gr}_{k_\Sigma}^W V_\Sigma$ is a CY variation with Hodge bundle $F^m \operatorname{gr}_{k_\Sigma}^W V_\Sigma$.

- We denote by $V_\Sigma \to V_\Sigma^{\min}$ the CY-minimal quotient of V_Σ and by V_Σ^{tr} the transcendental part of V_Σ in the sense of Definition 2.9. Note that V_Σ^{tr} is also the transcendental part of $\mathrm{gr}^W_{k_\Sigma} V_\Sigma$. Note also that even on the stratum $\Sigma = \varnothing$ where $X_\varnothing = X \backslash D$ and $V_\varnothing = V$, the transcendental part $V_\varnothing^{\mathrm{tr}}$ and the CY-minimal quotient $V_\varnothing^{\min}$ (which is equal to $V_\varnothing^{\mathrm{tr}}$) may be strictly smaller than V.
- For a single point $x \in X$, there is a unique X_Σ containing x and we define $V^{\min}(x)$ (resp. $V^{\mathrm{tr}}(x)$) as the CY-minimal quotient (resp. the transcendental part) of the mixed Hodge structure $V_{\Sigma,x}$. Note that at a very general point $x \in X_\Sigma(\mathbb{C})$ we simply have $V^{\mathrm{tr}}(x) = V^{\mathrm{tr}}_{\Sigma,x}$ and $V^{\min}(x) = V^{\min}_{\Sigma,x}$.
- The restriction of the Schmid extension of the Hodge bundle $F^m V_{\mathcal{O}}$ to $\overline{X_\Sigma}$ is naturally identified with the Schmid extension of the Hodge bundle of each of the following:
$$\mathrm{gr}^{W(\Sigma)}_{k_\Sigma} V|_{X_\Sigma}$$
$$V_\Sigma$$
$$\mathrm{gr}^W_{\geq k_\Sigma} V_\Sigma \coloneqq V_\Sigma / W_{k_\Sigma - 1} V_\Sigma$$
$$\mathrm{gr}^W_{k_\Sigma} V_\Sigma$$
$$V_\Sigma^{\mathrm{tr}}.$$
Indeed, all of the above constructions are compatible on the level of filtered logarithmic flat vector bundles.
- After choosing a very general basepoint $x_\Sigma \in X_\Sigma$, we thereby obtain a period map
$$\phi_\Sigma^{\mathrm{tr}} : \widetilde{X_\Sigma}^{\mathcal{V}^{\mathrm{tr}}_{\Sigma,\mathbb{Q}}} \to \check{\mathbb{D}}(V^{\mathrm{tr}}_{\Sigma,\mathbb{C},x_\Sigma})^{\mathrm{an}}$$
associated to V_Σ^{tr}, where $\check{\mathbb{D}}(V^{\mathrm{tr}}_{\Sigma,\mathbb{C},x_\Sigma})$ is a flag variety of filtrations on $V^{\mathrm{tr}}_{\Sigma,\mathbb{C},x_\Sigma}$.
- When V arises from a general variation ${}_{\mathrm{orig}}V$ as in Definition 2.11, and assuming the local monodromy of ${}_{\mathrm{orig}}V$ is unipotent, we further define ${}_{\mathrm{orig}}V_\Sigma^{\mathrm{gr}}$ to be the graded polarizable integral mixed variation with underlying local system ${}_{\mathrm{orig}}V^{\mathrm{gr}}_{\Sigma,\mathbb{Z}} = \bigoplus_k \mathrm{gr}_k^{W(\Sigma)} {}_{\mathrm{orig}}V_{\mathbb{Z}}|_{X_\Sigma}$. For $x \in X_\Sigma$ we take ${}_{\mathrm{orig}}V^{\mathrm{gr}}(x) = {}_{\mathrm{orig}}V^{\mathrm{gr}}_{\Sigma,x}$.
- We let $V^\vee$ denote the dual variation to V. We may canonically identify
$$V(\Sigma)_{\mathbb{Q}}|^\vee_{\mathfrak{T}^*(\Sigma)} \coloneqq \ker\left(\oplus_{E\in\Sigma} N_E^\vee : V_{\mathbb{Q}}^\vee|_{\mathfrak{T}^*(\Sigma)} \to \bigoplus_{E\in\Sigma} V_{\mathbb{Q}}^\vee|_{\mathfrak{T}^*(\Sigma)}\right) \subset j_*(V_{\mathbb{Q}}^\vee)|_{\mathfrak{T}^*(\Sigma)}$$
and thus $V(\Sigma)^\vee_{\mathbb{Q}} \subset j_*(V^\vee_{\mathbb{Q}})|_{\mathfrak{T}(\Sigma)}$.
- Moreover, we have a subvariation $V_\Sigma^{\min,\vee} \coloneqq (V_\Sigma^{\min})^\vee \subset V_\Sigma^\vee$, which is the smallest subvariation which pairs non-trivially with $\mathrm{gr}^m_F V_\Sigma$, and likewise $V_\Sigma^{\mathrm{tr},\vee}$ is the highest graded piece of $V_\Sigma^{\min,\vee}$. Note that the Hodge bundle of $V_\Sigma^{\mathrm{tr},\vee}$ is $F^{m-k_\Sigma} V_\Sigma^{\mathrm{tr},\vee}$.

Lemma 2.12. *(1) The local system $V^{\min,\vee}_{\Sigma,\mathbb{Q}}$ is the restriction of a local system $V^{\min,\vee}(\Sigma)_{\mathbb{Q}}$ on $\mathfrak{T}(\Sigma)$, which is naturally a subsheaf of $\mathcal{V}\,|_{\mathfrak{T}(\Sigma)}$ consisting of flat sections. The fibers of $V^{\min,\vee}(\Sigma)_{\mathbb{Q},x}$ at a point $x \in \mathfrak{T}^*(\Sigma)$ consist of all elements $s_x \in \mathcal{V}_x^\vee$ which extend to a flat section s along any contractible subset $U \subset \mathfrak{T}(\Sigma)$ whose restriction to X_Σ lands in $V^{\min,\vee}_{\Sigma,\mathbb{Q}}$. Moreover, it is sufficient to check this condition for a single contractible U which non-trivially intersects X_Σ.*

(2) If $\Sigma \subset \Sigma'$, we have the containments $V^{\min,\vee}(\Sigma')_{\mathbb{Q}} \subset V^{\min,\vee}(\Sigma)_{\mathbb{Q}}$ within $\mathfrak{T}(\Sigma) \cap \mathfrak{T}(\Sigma')$.

Proof. For (1), first note that $V^{\min,\vee}_{\Sigma,\mathbb{Q}}$ extends as a subsheaf of $j_* V^\vee_{\mathbb{Q}}\,|_{\mathfrak{T}(\Sigma)}$ by Lemma 2.7, and hence as a subsheaf of $(\mathcal{V}^\vee)^{\mathrm{an}}\,|_{\mathfrak{T}(\Sigma)}$ since $j_* V^\vee_{\mathbb{Q}} \subset (\mathcal{V}^\vee)^{\mathrm{an}}$. Moreover, the image of any section of $j_* V^\vee_{\mathbb{Q}}$ is flat. Finally, it is clear by our construction that any element of $V^{\min,\vee}(\Sigma)_{\mathbb{Q},x}$ extends along any contractible set U to a section, and the fiber along $X_\Sigma \cap U$ will land in $V^{\min,\vee}_{\Sigma,\mathbb{Q}}$. To see that it is sufficient to consider a single contractible open set U which intersects X_Σ, note that any such U gives a canonical identification of stalks which preserves the restrictions of global flat sections.

For (2), first note that the Deligne extension of $V^{\min,\vee}_{\Sigma,\mathbb{Q}}$ along $X_{\Sigma'}$ is naturally a flat subbundle of $\mathcal{V}^\vee|_{\overline{X}_\Sigma}$ whose restriction to $X_{\Sigma'}$ underlies a rational subvariation of Hodge structures with $\mathrm{gr}_F^{-m} \neq 0$ hence contains $V^{\min,\vee}_{\Sigma',\mathbb{Q}}$. It is clear that $V(\Sigma')^\vee_{\mathbb{Q}} \subset V(\Sigma)^\vee_{\mathbb{Q}}$ on $\mathfrak{T}(\Sigma) \cap \mathfrak{T}(\Sigma')$. Now take a contractible $U \subset \mathfrak{T}(\Sigma')$ which intersects $X_{\Sigma'}$ (and therefore also X_Σ). Let $x \in \mathfrak{T}(\Sigma) \cap U \backslash D$ be a point whose path-component in $\mathfrak{T}(\Sigma) \cap U$ meets X_Σ. By (1) any element of $V^{\min,\vee}(\Sigma')_{\mathbb{Q},x}$ extends to a flat global section s over U. By the above, the restriction $s\,|_{X_\Sigma}$ is contained in $V^{\min,\vee}_{\Sigma,\mathbb{Q}}$, so by (1) again the conclusion follows. □

2.6. **The CY-minimal quotient.** Let (X, D) be a proper strictly log smooth algebraic space and $V = (V_\mathbb{Z}, F^\bullet V_\mathcal{O})$ a polarizable integral pure CY variation on $X\backslash D$ with unipotent local monodromy. By [BBT23a], for any $\Sigma \subset \pi_0(D^{\mathrm{reg}})$ we may consider the factorization of the period map of V^{tr}_Σ

$$X^{\mathrm{def}}_\Sigma \xrightarrow{f^{\mathrm{def}}_\Sigma} Y^{\mathrm{def}}_\Sigma \xrightarrow{\psi_\Sigma} \Gamma_\Sigma\backslash\mathbb{D}_\Sigma$$

where f_Σ is dominant with geometrically connected general fiber, Y_Σ is normal, and ψ_Σ is finite. Note that f_Σ does not depend on Γ_Σ as it is the Stein factorization of any relative compactification of any period map associated to V^{tr}_Σ. If the monodromy of V^{tr}_Σ is neat, then V^{tr}_Σ is pulled back from a variation on Y_Σ, say $V^{\mathrm{tr}}_\Sigma = f^*_\Sigma U$. Recall that V^{tr}_Σ is the lowest weight piece of the CY-minimal quotient $V_\Sigma \to V^{\min}_\Sigma$, that is, $V^{\mathrm{tr}}_\Sigma = W_{k_\Sigma} V^{\min}_\Sigma$.

Lemma 2.13. *Assume the monodromy of $V_\mathbb{Z}$ is neat. There is a dense open subset $Y^\circ_\Sigma \subset Y_\Sigma$ such that, setting $X^\circ_\Sigma = f^{-1}_\Sigma(Y^\circ_\Sigma)$, we have that $V^{\min}_\Sigma|_{X^\circ_\Sigma}$ is pulled back from a local system on Y°_Σ.*

Proof. Let $Y^\circ_\Sigma \subset Y_\Sigma$ be a dense open set for which the fibers of f_Σ are smooth and f_Σ is a topological fibration. As the fibers are connected, it suffices to show $V^{\min,\vee}_\Sigma$ is trivial on a very general fiber Z of f_Σ over Y°_Σ. The connection yields a morphism $\mathrm{gr}_F^{m-k_\Sigma} V^{\min,\vee}_\Sigma \to \mathrm{gr}_F^{m-k_\Sigma-1} V^{\min,\vee}_\Sigma \otimes \Omega_Z$ and we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{gr}_F^{m-k_\Sigma} W_{-k_\Sigma-1} V^{\min,\vee} & \longrightarrow & \mathrm{gr}_F^{m-k_\Sigma} V^{\min,\vee}_\Sigma & \xrightarrow{\cong} & \mathrm{gr}_F^{m-k_\Sigma} V^{\mathrm{tr},\vee}_\Sigma \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle 0} \\
0 = \mathrm{gr}_F^{m-k_\Sigma-1} W_{-k_\Sigma-1} V^{\min,\vee}_\Sigma \otimes \Omega_Z & \longrightarrow & \mathrm{gr}_F^{m-k_\Sigma-1} V^{\min,\vee} \otimes \Omega_Z & \longrightarrow & \mathrm{gr}_F^{m-k_\Sigma-1} V^{\mathrm{tr},\vee}_\Sigma \otimes \Omega_Z
\end{array}$$

where the bottom left vanishing follows from the fact that $F^m \mathrm{gr}^W_{>k_\Sigma} V^{\min}_\Sigma = 0$, so $V^{\min}_\Sigma$ has no Hodge weight (p, q) piece for $p \geq m$ and $p + q > k_\Sigma$, hence $V^{\min,\vee}_\Sigma$ has no Hodge weight (p, q) piece for $q \leq -m$ and $p + q < -k_\Sigma$, and in particular for $p = m - k_\Sigma - 1$. Thus, the middle vertical morphism vanishes, and $F^{m-k_\Sigma} V^{\min,\vee}_\Sigma$ is flat on Z. It is therefore isomorphic as a flat bundle to $\mathrm{gr}_F^{m-k_\Sigma} V^{\mathrm{tr},\vee}_\Sigma$, hence has trivial monodromy. By the theorem of the fixed part [And92], the fixed part $H^0(Z, V^{\min,\vee}_\Sigma)$ comes from a sub-variation of mixed Hodge structures which contains the Hodge bundle, hence $(V_\Sigma|_Z)^{\min,\vee}$ has trivial monodromy. Since Z is a very general fiber, we have $(V_\Sigma|_Z)^{\min,\vee} = V^{\min,\vee}_\Sigma|_Z$, and this proves the claim. □

Corollary 2.14. *Assume the monodromy of $V_{\mathbb{Z}}$ is neat. For each stratum Σ we have a commutative diagram*

$$\begin{tikzcd}
\widetilde{\mathfrak{T}^\circ(\Sigma)}^{V^{\mathrm{tr}}_{\Sigma,\mathbb{Q}}|_{\mathfrak{T}^\circ(\Sigma)}} \arrow[rr, "\rho(\Sigma)"] & & \mathbb{P}(V^{\min}_{\Sigma,\mathbb{C},x(\Sigma)})^{\mathrm{an}} \\
\widetilde{X^\circ_\Sigma}^{V^{\mathrm{tr}}_{\Sigma,\mathbb{Q}}|_{X^\circ_\Sigma}} \arrow[u] \arrow[r, "\phi^{\mathrm{tr}}_\Sigma"] & \check{\mathbb{D}}(V^{\mathrm{tr}}_{\Sigma,\mathbb{C},x_\Sigma})^{\mathrm{an}} \arrow[r] & \mathbb{P}(V^{\mathrm{tr}}_{\Sigma,\mathbb{C},x_\Sigma})^{\mathrm{an}} \arrow[u]
\end{tikzcd} \tag{2.2}$$

where $\mathfrak{T}^\circ(\Sigma)$ is a DR-neighborhood of X°_Σ as in Lemma 2.13. Here, the bottom right horizontal map is the forgetful map which only remembers the Hodge line, the right vertical map is obtained from the inclusion $V^{\mathrm{tr}}_\Sigma \hookrightarrow V^{\min}_\Sigma$, and the top map is obtained by taking the image of the Hodge bundle $F^m\mathcal{V}$ under the natural morphism of logarithmic flat bundles

$$\mathcal{V}|_{\mathfrak{T}^\circ(\Sigma)} \to V^{\min}(\Sigma)_{\mathcal{O}}|_{\mathfrak{T}^\circ(\Sigma)}$$

where $V^{\min}(\Sigma)_{\mathcal{O}}|_{\mathfrak{T}^\circ(\Sigma)}$ is the flat bundle associated to the extension of the local system $V^{\min}_{\Sigma,\mathbb{Q}}|_{X^\circ_\Sigma}$ to $\mathfrak{T}^\circ(\Sigma)$.

Proof. The above composition is full rank on $F^m\mathcal{V}$ in restriction to X_Σ and this is an open condition. □

Remark 2.15. The analysis of the differential in Lemma 2.13 (in the case of the Griffiths bundle) is related to the "infinitesimal period relation" of [GGR25, §2,4]. The flat connection on the Griffiths bundle in tubular neighborhoods of numerically Hodge-trivial subvarieties constructed therein is obtained by the trivializing sections pulled back via the map $\rho(\Sigma)$ of Corollary 2.14.

2.7. **Integrability.** In the following for a stratum $\Sigma \subset \pi_0(D^{\mathrm{reg}})$ we define $D_\Sigma := \bigcup_{E\notin\Sigma} \overline{E}|_{\overline{X_\Sigma}}$ to be the natural log smooth divisor of $\overline{X_\Sigma}$.

Lemma 2.16. *Let (X, D) be a proper log smooth algebraic space and $V = (V_{\mathbb{C}}, F^\bullet V_{\mathcal{O}})$ a CY polarizable complex variation of Hodge structures on $X\backslash D$ with unipotent local monodromy. Let $L = F^m\mathcal{V}$ be the Hodge bundle, that is, the Schmid extension of $F^m V_{\mathcal{O}}$.*

(1) L is nef.
(2) The Schmid extension of any power $(F^m V_{\mathcal{O}})^k$ is naturally identified with the same power L^k of the Hodge bundle.
(3) For any proper log smooth (Y, D_Y) and morphism $g\colon (Y, D_Y) \to (\overline{X_\Sigma}, D_\Sigma)$, the pullback of the Hodge bundle g^*L is naturally identified with the Hodge bundle of the pullback $g|^*_{Y\backslash D_Y} V^{\mathrm{tr}}_\Sigma$.

Proof. Part (1) is as in [BBT23a, Lemma 6.15]. Parts (2) and (3) follow from the fact that the Deligne extension is functorial with respect to pullbacks and tensor operations if the local monodromy is unipotent. □

Definition 2.17. Let (X, D) be a proper log smooth algebraic space and $V = (V_{\mathbb{C}}, F^\bullet V_{\mathcal{O}})$ a polarizable complex CY variation of Hodge structures on $X\backslash D$ with unipotent local monodromy with Deligne/Schmid extension $(\mathcal{V}, F^\bullet\mathcal{V})$. We say the Hodge bundle $L = F^m\mathcal{V}$ is *integrable* if, after replacing (X, D) with a strictly log smooth modification, for any irreducible proper strictly log smooth (Y, D_Y) with a morphism $g\colon (Y, D_Y) \to (\overline{X_\Sigma}, D_\Sigma)$ which is generically finite onto its image, g^*L is big whenever either of the following equivalent conditions is satisfied:

(1) The Griffiths bundle of the transcendental part of the pullback $(g|^*_{Y\backslash D_Y} V^{\mathrm{tr}}_\Sigma)^{\mathrm{tr}}$ is big.
(2) The period map of the transcendental part of the pullback $(g|^*_{Y\backslash D_Y} V^{\mathrm{tr}}_\Sigma)^{\mathrm{tr}}$ on $Y\backslash D_Y$ is generically immersive.

Concretely, this means that if g^*L is not big, then some subvariation of the pullback $g|_{Y\backslash D_Y}^* V_\Sigma^{\mathrm{tr}}$ containing the Hodge bundle is isotrivial on a curve through the generic point of Y.

Remark 2.18. It is proven in [BBT23a] that the Griffiths bundle is semiample on $X\backslash D$. The same is proven for the Hodge bundle of a CY variation whenever the Kodaira–Spencer map on the Hodge bundle is immersive. The strategy of Section 4 can be used to prove the Hodge bundle of a CY variation is semiample on $X\backslash D$ if it is integrable (on $X\backslash D$) in the above sense.

Lemma 2.19. *Let (X, D) be a proper log smooth algebraic space and $({}_{\mathrm{orig}}V_{\mathbb{C}}, F^\bullet{}_{\mathrm{orig}}V_{\mathcal{O}})$ a polarizable complex variation of Hodge structures on $X\backslash D$ with Deligne/Schmid extension $({}_{\mathrm{orig}}\mathcal{V}, F^\bullet{}_{\mathrm{orig}}\mathcal{V})$. Set*

$$V \coloneqq \bigotimes_p \bigwedge^{\mathrm{rk}\, F^p{}_{\mathrm{orig}}V_{\mathcal{O}}} {}_{\mathrm{orig}}V.$$

Then:

1. *For any subvariety $Y \subset \overline{X_\Sigma}$ with $Y_\Sigma \coloneqq Y \cap X_\Sigma \neq \varnothing$, the following are equivalent:*
   1. *The Hodge bundle of V is big in restriction to Y.*
   2. *The Griffiths bundle of V is big in restriction to Y.*
2. *The Hodge bundle of V is integrable.*

Proof. For (1), $(a) \Rightarrow (b)$ since the Griffiths bundle is the sum of the Hodge bundle and a semipositive line bundle. For the converse, recall that the Griffiths bundle is ample on the image of a period map, by [BBT23a]. Thus, if the Hodge bundle of V (which is the Griffiths bundle of ${}_{\mathrm{orig}}V$) is not big, then ${}_{\mathrm{orig}}V$ is isotrivial on a curve through the general point, as therefore is V, so the Griffiths bundle of V is not big. (2) is an immediate consequence, since if the Hodge bundle of V is not big in restriction to Y, $({}_{\mathrm{orig}}V)_\Sigma$ is again isotrivial on a curve through the generic point of Y, as therefore is a part of V_Σ which contains the Hodge bundle. $\square$

2.8. **Unpolarized Hodge structures.** Let $\mathbf{M}$ be the Mumford–Tate group of a polarizable integral pure Hodge structure $(V_{\mathbb{Z}}, F^\bullet V_{\mathbb{C}})$, $\mathbb{D}$ the $\mathbf{M}(\mathbb{R})$-orbit of $F^\bullet V_{\mathbb{C}}$ in the flag variety of filtrations on the vector space $V_{\mathbb{C}}$, and $\mathbf{M}(\mathbb{Z}) \subset \mathbf{M}(\mathbb{Q})$ the subgroup stabilizing the lattice induced by $V_{\mathbb{Z}}$ in a chosen faithful representation of $\mathbf{M}$. The following generalizes a result of Narasimhan–Nori [NN81] in the case of abelian varieties and Huybrechts [Huy18, Cor 1.8] in the case of hyperkähler varieties.

Lemma 2.20. *For any $x \in \mathbf{M}(\mathbb{Z})\backslash\mathbb{D}$, there are finitely many points $x' \in \mathbf{M}(\mathbb{Z})\backslash\mathbb{D}$ for which the associated integral Hodge structures $V(x), V(x')$ are isomorphic.*

Proof. In [NN81], this claim is proven for Abelian varieties, and the same proof works in this more general context. We reproduce the proof for the ease of the reader.

First, note that $\mathbf{M}(\mathbb{Z})\backslash\mathbb{D}$ has a finite map to a usual period space $\mathrm{Aut}(V_{\mathbb{Z}}, Q)\backslash\mathbb{D}'$ corresponding to a choice of polarized lattice $(V_{\mathbb{Z}}, Q)$, and so it is enough to work with the latter space.

Next, consider the algebra $B = \mathrm{End}(V(x))$ of unpolarized Hodge endomorphisms. Since polarizable Hodge structures are a semisimple category, it follows that $B_{\mathbb{Q}}$ is a semisimple algebra over $\mathbb{Q}$. Moreover, the polarization gives an involution θ of $B_{\mathbb{Q}}$. Letting G be the algebraic group $B^\times$ we see that θ gives an involution $G \to G^{\mathrm{op}}$. We shall prove that $V(x)$ admits only finitely many orbits of polarizations of discriminant $\mathrm{disc}(Q)$ for the action of $G(\mathbb{Z})$, which will prove the lemma.

Let P be the set of polarizations of $V(x)$. There is a natural injection $\iota : P \to B_{\mathbb{Q}}$ given by $\iota(Q') \coloneqq \phi_Q^{-1} \circ \phi_{Q'}$ where $\phi_{Q'} : V(x) \to V(x)^\vee$. Note the $\iota(P)$ is contained in a minimal lattice $F \subset B^{\theta=1}$. There is a natural action of G on $B^{\theta=1}$ given by $\pi(g)s \coloneqq \theta(g^{-1}) \circ s \circ g^{-1}$ for which ι is equivariant, and for which $G(\mathbb{Z})$ preserves F.

Finally, let $F_1 \coloneqq \{f \in B_{\mathbb{C}}^{\theta=1} \mid \deg(f) = 1\}$. As in [NN81, Lemma 3.1][7] the orbits of $G_{\mathbb{C}}$ on F_1 are finite in number and closed. The result now follows by [Bor62, Thm 6.9]. □

2.9. **Combinatorial monodromy.** For any proper algebraic space X, a torsion line bundle L has a canonical flat connection: if $L^N \cong \mathcal{O}_X$, then the local flat sections are those sections s for which s^N extends to a global section. Equivalently, there is a finite étale cover $\pi : X' \to X$ for which $\pi^* L \cong \mathcal{O}_{X'}$, and the flat connection on L is inherited from the trivial connection on X' whose flat sections are global sections.

If X is now a nodal curve with normalization $\nu_X : X' \to X$ and L a line bundle for which $\nu_X^* L$ is torsion, L has a canonical flat connection whose local flat sections are sections s whose restriction to X^{reg} are flat with respect to the above connection on $L|_{X^{\mathrm{reg}}}$, or equivalently those for which $\nu_X^* s$ is flat.

Definition 2.21. Let X be a proper algebraic space with a line bundle L. We say L has torsion combinatorial monodromy if for every proper nodal curve C and morphism $g : C \to X$ for which $(g \circ \nu_C)^* L$ is torsion, the canonical flat connection on $g^* L$ has torsion monodromy.

2.9.1. *Torsion combinatorial monodromy of the Griffiths bundle.* Thanks to a result of [GGR25], the Griffiths bundle always has torsion combinatorial monodromy.

Theorem 2.22 (Green–Griffiths–Robles [GGR25, Theorem 5.22])**.** *Let (X, D) be a proper log smooth algebraic space and ${}_{\mathrm{orig}}V = ({}_{\mathrm{orig}}V_{\mathbb{Z}}, F^{\bullet}_{\mathrm{orig}} V_{\mathcal{O}})$ a polarizable integral pure variation of Hodge structures on $X \backslash D$ with unipotent local monodromy. Then the Griffiths bundle has torsion combinatorial monodromy.*

We give a slight generalization below. We say the Hodge bundle of a CY variation has norm one combinatorial monodromy if the monodromy of the canonical connection on any Hodge degree 0 nodal curve acts by a character of complex norm one.

Lemma 2.23. *Let (X, D) be a proper log smooth algebraic space and $(V_{\mathbb{Z}}, F^{\bullet} V_{\mathcal{O}})$ a polarizable integral pure CY variation of Hodge structures on $X \backslash D$ with unipotent local monodromy. If the Hodge bundle has norm one combinatorial monodromy, then it has torsion combinatorial monodromy.*

Proof. For any connected nodal curve $g : C \to X$ with Hodge degree 0, the pointwise transcendental parts $V^{\mathrm{tr}}(g(c))$ form an isotrivial simple not-necessarily-polarizable integral pure variation of Hodge structures $V^{\mathrm{tr}}(C)$ which is polarizable on each irreducible component. It suffices to show $V^{\mathrm{tr}}(C)$ is polarizable, which is a consequence of the following:

Claim 2.24. *Let $U = (U_{\mathbb{Z}}, F^{\bullet} U_{\mathbb{C}})$ be a simple polarizable integral pure CY Hodge structure with deepest Hodge filtration piece $F^m U_{\mathbb{C}}$ and γ a Hodge automorphism of $U_{\mathbb{Z}}$ which acts with norm one eigenvalues on $F^m U_{\mathbb{C}}$. Then γ preserves any polarization q on U; in particular, it is torsion.*

Proof. Following the proof of [GGR25], consider

$$q - \gamma^* q : U_{\mathbb{Q}} \to U_{\mathbb{Q}}^{\vee}(-w)$$

where w is the weight of U. For $v \in F^m U_{\mathbb{C}}$, if $\gamma v = \alpha v$ with $|\alpha|^2 = 1$ then we have $q(v, \overline{v}) = |\alpha|^2 q(v, \overline{v}) = q(\gamma v, \gamma \overline{v})$. On the other hand, for $u \in U_{\mathbb{C}}$ with no Hodge component in $\overline{F^m U_{\mathbb{C}}}$, the same is true for γu, and thus $q(v, u) = 0 = q(\gamma v, \gamma u)$. Thus, $v \in \ker(q - \gamma^* q)$. But $\ker(q - \gamma^* q) \subset U_{\mathbb{Q}}$ is then a nonzero sub-$\mathbb{Q}$-Hodge structure of $U_{\mathbb{Q}}$ which must be all of $U_{\mathbb{Q}}$ since $U_{\mathbb{Q}}$ is simple. □

□

[7] This lemma uses only that $(B_{\mathbb{Q}}, \theta)$ is an involutive semisimple algebra.

Theorem 2.22 follows from Lemma 2.23 as follows. Let L be the Griffiths bundle of ${}_{\mathrm{orig}}V$. As in the proof of Lemma 2.23, for any connected Griffiths degree 0 curve $g : C \to X$, we obtain a well-defined isotrivial integral mixed variation ${}_{\mathrm{orig}}V^{\mathrm{gr}}(C)$ which is graded polarizable on each irreducible component. Set $E_k := \mathrm{gr}^W_k\, {}_{\mathrm{orig}}V^{\mathrm{gr}}(C)$. The polarization q of ${}_{\mathrm{orig}}V$ gives a *global* isomorphism $E_{w+k} \cong E^\vee_{w-k}(-w)$ where w is the weight of ${}_{\mathrm{orig}}V$. We also have global isomorphisms $F^pE_k \cong \overline{F^{k-p+1}E_k}$ for all p and k. Thus,

$$F^pE_{w+k} \cong (E_{w-k}/F^{w-p+1}E_{w-k})^\vee \cong \overline{F^{p-k}E_{w-k}}^\vee.$$

If L_k is the Griffiths bundle of E_k, then it follows that $L_{w+k} \cong \overline{L_{w-k}}^\vee$, and since $g^*L = \bigotimes_k L_k$, we have $g^*L \cong \overline{g^*L}^\vee$. Thus, g^*L is conjugate-self dual, and hence the induced character of the $\pi_1(C)$ action on L is norm 1.

For later, we record the following consequence of the argument:

Lemma 2.25. *For any $x \in X$, the Griffiths line of ${}_{\mathrm{orig}}V^{\mathrm{gr}}(x)$ is canonically conjugate self-dual.*

3. Quotient spaces

3.1. **Equivalence relations.** Let X be a proper algebraic space, and let $R \subset X(\mathbb{C}) \times X(\mathbb{C})$ be a closed reflexive symmetric constructible relation and let $p_i : R \to X(\mathbb{C})$ be the two projections. Then

$$R^+ := (p_1 \times p_2)(R\, {}_{p_2}\!\times_{p_1} R) \subset X(\mathbb{C}) \times X(\mathbb{C})$$

is also a closed reflexive symmetric constructible relation, which on the level of points is defined by $x \sim_+ y$ if $x \sim z \sim y$ for some z. Note that if for some constructible $U \subset X(\mathbb{C})$ we have

$$R^+ \cap (U \times X(\mathbb{C})) = R \cap (U \times X(\mathbb{C}))$$

then we also have

$$(R^+)^+ \cap (U \times X(\mathbb{C})) = R \cap (U \times X(\mathbb{C})).$$

In this case, we say the R-related classes of U are stable. There is a maximal constructible subset $U \subset X(\mathbb{C})$ whose R-related classes are stable, given by $U = X(\mathbb{C})\backslash p_1^+(R^+\backslash R)$.

The equivalence relation generated by a closed, reflexive, symmetric, constructible relation R is obtained by setting $R_0 = R$ and letting $R_{j+1} = (R_j)^+$. Then R_j is the relation of being connected by a chain of 2^j R-equivalences, and $R^e := \bigcup_j R_j$ is the smallest equivalence relation containing R. By the above, there is a maximal constructible $U_j \subset X(\mathbb{C})$ whose R_j-related classes are stable, and the U_j form an increasing sequence of subsets of X.

Lemma 3.1. *Let X be a proper algebraic space and $R \subset X(\mathbb{C}) \times X(\mathbb{C})$ a closed reflexive symmetric constructible relation. Then there is a closed constructible subset $\Delta \subset X(\mathbb{C})$ such that:*

1. *Every $x \in X(\mathbb{C})$ with non-constructible R^e-equivalence class is contained in Δ.*
2. *The set of points $x \in \Delta$ with constructible R^e-equivalence class is contained in a countable union of nowhere dense constructible subsets of Δ.*

Proof. By the above, the R^e-equivalence class of x is a union of the R_j-related classes C_j of x, which form an increasing sequence of constructible subsets. If $\bigcup_j C_j$ is constructible, then it must stabilize, $C_j = \bigcup_j C_j$, for some j. In the above notation, let $\Delta_j = X(\mathbb{C})\backslash U_j$. Then the Δ_j form a decreasing sequence of constructible subsets, hence the closures $\overline{\Delta}_j$ stabilize to Δ, and $\Delta \cap \bigcup_j U_j$ is a countable union of nowhere dense constructible subsets of Δ. □

Remark 3.2. For any algebraic space X and proper constructible equivalence relation $R \subset X(\mathbb{C}) \times X(\mathbb{C})$, the quotient $X(\mathbb{C})/R$ exists in the category of definable topological spaces and we will always mean it as such. Note that $X(\mathbb{C})/R$ can also be endowed with a Zariski topology, which is the quotient topology obtained by endowing $X(\mathbb{C})$ with the Zariski topology. If $q : X(\mathbb{C}) \to X(\mathbb{C})/R$ is the quotient, then for any closed constructible $Z \subset X(\mathbb{C})$ its saturation $q^{-1}(q(Z)) \subset X(\mathbb{C})$ is closed constructible, either by definable Chow or because it is identified with $p_2(R \cap p_1^{-1}(Z))$.

3.2. **Hodge-theoretic equivalence relations.** Let (X, D) be a proper log smooth algebraic space and $(V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ a polarizable integral pure CY variation on $X\backslash D$ with unipotent local monodromy and integrable Hodge bundle L. There are several natural equivalence relations on $X(\mathbb{C})$.

3.2.1. We define $R_{\mathrm{tr}} \subset X(\mathbb{C}) \times X(\mathbb{C})$ to be the equivalence relation defined by $x \sim_{\mathrm{tr}} y$ if $V^{\mathrm{tr}}(x) \cong V^{\mathrm{tr}}(y)$ as (unpolarized) integral pure Hodge structures.

3.2.2. We define $R_{\mathrm{curve}} \subset X(\mathbb{C}) \times X(\mathbb{C})$ to be the equivalence relation defined by $x \sim_{\mathrm{curve}} y$ if there is a proper connected curve $g : C \to X$ with $x, y \in g(C)$ for which $\deg g^*L = 0$. For any irreducible component C_0 of such a curve, and X_Σ the unique stratum containing the generic point of C_0, the transcendental part of V^{tr}_Σ restricted to $C_0 \cap X_\Sigma$ is isotrivial. It follows that the pointwise transcendental parts $V^{\mathrm{tr}}(g(c))$ for $c \in C$ form an isotrivial not-necessarily-polarizable variation of integral pure Hodge structures $V^{\mathrm{tr}}(C)$ over C which is polarizable in restriction to each irreducible component. In particular, $R_{\mathrm{curve}} \subset R_{\mathrm{tr}}$.

Lemma 3.3. *The Hodge bundle $L = F^m\mathcal{V}$ has torsion combinatorial monodromy if and only if for any proper curve $g : C \to X$ with $\deg g^*L = 0$, the isotrivial variation $V^{\mathrm{tr}}(C)$ has finite monodromy.*

Proof. Let $c \in C$ be a basepoint. Since $V^{\mathrm{tr}}(g(c)) = V^{\mathrm{tr}}(C)_c$ is simple, a Hodge automorphism is trivial if and only if it is trivial on the component $\mathrm{gr}^m_F V^{\mathrm{tr}}(C)_c$. □

Lemma 3.4.

1. *For any connected constructible subset Z of a R_{tr}-equivalence class, the closure $\overline{Z}$ is contained in an R_{curve}-equivalence class.*
2. *If a R_{tr}-equivalence class is constructible, then it is closed and its connected components are R_{curve}-equivalence classes.*
3. *For any point $x \in X(\mathbb{C})$ for which $V^{\mathrm{tr}}(x)$ is maximal rank, the R_{tr}-equivalence class of x is constructible.*
4. *For any closed reflexive symmetric constructible relation $R \subset X(\mathbb{C}) \times X(\mathbb{C})$ for which $R_{\mathrm{curve}} \subset R^e \subset R_{\mathrm{tr}}$, R^e is a closed constructible equivalence relation whose equivalence classes are finite unions of R_{curve}-equivalence classes.*

Proof. For (1), if a R_{tr}-equivalence class contains a constructible set Z, then for any proper connected curve $C \subset X$ whose generic point is contained in Z, $V^{\mathrm{tr}}(C)$ as above is isotrivial, so C is contained in a R_{curve}-equivalence class, hence $C \subset Z$. Thus $\overline{Z}$ is contained in a R_{curve}-equivalence class. Part (2) follows immediately from (1).

For (3), let $x \in X(\mathbb{C})$ be such a point and Z its R_{tr}-equivalence class. Then for any $z \in Z \cap X_\Sigma$, $V^{\mathrm{tr}}(x) = V^{\mathrm{tr}}_\Sigma$. Applying Lemma 2.20, $Z \cap X_\Sigma$ is constructible, hence Z is.

For (4), by Lemma 3.1 there is a closed constructible $\Delta \subset X(\mathbb{C})$ such that outside a countable union of nowhere dense constructible subsets $\Xi \subset \Delta$, each R-equivalence class is non-constructible. However applying (3) to (a resolution of) each irreducible component Δ_0, each $R_{\mathrm{tr}}|_{\Delta_0}$-equivalence class in Δ_0 of maximal rank transcendental part is constructible, as therefore is any $R_{\mathrm{tr}}|_\Delta$-equivalence class in Δ of maximal rank transcendental part. If Δ were nonempty, there would then be a point $x \in \Delta\backslash\Xi$ of maximal rank transcendental part, hence constructible $R_{\mathrm{tr}}|_\Delta$-equivalence class Z. By (2), Z is partitioned into finitely many constructible R_{curve}-equivalence classes, hence also into finitely many constructible R^e-equivalence classes. This is a contradiction, so $\Delta = \varnothing$. □

3.3. **Properties of stratifications.** Let (X, D) be a proper strictly log smooth algebraic space and $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ a polarizable integral pure CY variation on $X\backslash D$ with unipotent local monodromy. As in Section 2.6, again consider the period map of V_Σ^{tr}

$$X_\Sigma^{\mathrm{def}} \xrightarrow{f_\Sigma^{\mathrm{def}}} Y_\Sigma^{\mathrm{def}} \xrightarrow{\psi_\Sigma} \Gamma_\Sigma\backslash\mathbb{D}_\Sigma.$$

Note that some power of the Hodge bundle naturally descends to Y_Σ. Recall that by the Griffiths criterion, $f_\Sigma : X_\Sigma \to Y_\Sigma$ extends to a proper map $\breve{f}_\Sigma : \breve{X}_\Sigma \to Y_\Sigma$ where $\breve{X}_\Sigma$ is the union of strata obtained from $\overline{X_\Sigma}$ by deleting divisors E along which V_Σ^{tr} has nontrivial monodromy.

Property 3.5. Let (X, D) be a proper strictly log smooth algebraic space and $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ a polarizable integral pure CY variation on $X\backslash D$ with unipotent local monodromy. Let R be an algebraic equivalence relation on $X(\mathbb{C})$ such that $R_{\mathrm{curve}} \subset R \subset R_{\mathrm{tr}}$. We define the following property of a boundary component $\Sigma \subset \pi_0(D^{\mathrm{reg}})$.

(B1) For any irreducible curve $C \subset X_\Sigma$ whose closure $\overline{C} \subset \overline{X_\Sigma}$ has degree 0 with respect to the Hodge bundle, C is contained in a fiber of f_Σ.

Under the assumption that the monodromy of $V_{\mathbb{Z}}$ is neat, we consider the following properties.

(B2) The open set $Y_\Sigma^\circ \subset Y_\Sigma$ from Lemma 2.13 is all of Y_Σ.

(B3)$_R$ For any other stratum $X_{\Sigma'}$, every irreducible component of $R\,|_{X_\Sigma \times X_{\Sigma'}}$ is surjective onto X_Σ.

We say (X, D) satisfies (B1), (B2), or (B3)$_R$ if every boundary stratum does so. For $R = R^e_{\mathrm{curve}}$ we simply write (B3) with no subscript.

(B1) is equivalent to the Hodge bundle being strictly nef on Y_Σ, i.e., on a log smooth compactification of a resolution, the extended Hodge bundle has positive degree on any curve meeting the interior. For the next lemma, observe that for proper log smooth algebraic spaces $(X, D), (X', D')$ and a morphism $\pi : X' \to X$ with $\pi^{-1}(D) \subset D'$, strata map to strata. Indeed, the inverse image of any closed stratum $\overline{X_\Sigma} = \bigcap_{E\in\Sigma} \overline{E}$ is a union of closed strata

$$\bigcap_{E\in\Sigma} \pi^{-1}(\overline{E}) = \bigcap_{E\in\Sigma} \bigcup_{\substack{E'\in\pi_0(D'^{\mathrm{reg}})\\ \pi(\overline{E}')\subset\overline{E}}} \overline{E}'.$$

Lemma 3.6. *Consider a relation R as above satisfying (B1-2). Let $R_1 = R \cap X_\Sigma \times X_{\Sigma'}$ and let $R_0 \subset R_1$ be an irreducible component.*

(1) If R_0 is not dominant over X_Σ, then it is not dominant over Y_Σ.

(2) If R_0 is dominant over X_Σ, then the complement of its projection $X_\Sigma\backslash\pi_{X_\Sigma}(R_0)$ is not dominant over Y_Σ.

Proof. Note that for a general $y \in Y_\Sigma$, the fiber $f_\Sigma^{-1}(y)$ is equidimensional of dimension $\dim X_\Sigma - \dim Y_\Sigma$. Since $R_{\mathrm{curve}} \subset R$, it follows that there is a relation $S \subset Y_\Sigma \times Y_{\Sigma'}$ such that R_1 is the pullback of S. By (B1), and since $R \subset R_{\mathrm{tr}}$ is an algebraic equivalence relation, it follows that S is quasifinite over Y_Σ. Since R_0 is an irreducible component of R_1 it follows that it is an irreducible component of $f_{\Sigma,\Sigma'}^{-1}(S_0)$ for some irreducible component $S_0 \subset S$.

For (1), assume that R_0 is dominant over Y_Σ. Then for a general $y \in Y_\Sigma$, for $(y, y') \in S_0$ we must have that the fiber $R_{0,(y,y')}$ of R_0 over (y, y') is an irreducible component of $f_\Sigma^{-1}(y)\times f_{\Sigma'}^{-1}(y')$, and hence dominates an irreducible component of $f_\Sigma^{-1}(y)$. Thus R_0 dominates a subset of X_Σ of dimension $\dim X_\Sigma$, and hence is dominant over X_Σ as desired.

For (2), assume that R_0 is dominant over X_Σ. Then it is also dominant over Y_Σ, and so just like the above it follows that $\pi_{X_\Sigma}(R_0)$ contains an irreducible component of $f_\Sigma^{-1}(y)$ for a generic $y \in Y_\Sigma$. Since X_Σ

is irreducible it follows that $\pi_{X_\Sigma}(R_0)$ in fact contains all of $f_\Sigma^{-1}(y)$ for a generic $y \in Y_\Sigma$, which implies the claim. $\square$

Lemma 3.7. *Let (X, D) be a proper log smooth algebraic space and $V = (V_\mathbb{Z}, F^\bullet V_\mathcal{O})$ a polarizable integral pure CY variation on $X \backslash D$ with unipotent local monodromy, and integrable Hodge bundle L.*

1. *Let (X', D') be a proper strictly log smooth algebraic space and $\pi : X' \to X$ a modification with $\pi^{-1}(D) \subset D'$. If X_Σ satisfies (B1) (resp. (B2)), the so does any stratum mapping to X_Σ.*
2. *There is a proper strictly log smooth algebraic space (X', D') and a modification $\pi : X' \to X$ with $\pi^{-1}(D) \subset D'$ such that (X', D') satisfies (B1). If the monodromy of $V_\mathbb{Z}$ is neat, then we may take (X', D') to satisfy (B2) and $(B3)_{R'}$, where R' is the pullback of R to X'.*

Proof. The first part is clear, since if $X'_{\Sigma'}$ maps to X_Σ, then $V'^{\mathrm{tr}}_{\Sigma'}$ and $V'^{\min}_{\Sigma'}$ are naturally the pullbacks of V^{tr}_Σ and $V^{\min}_\Sigma$.

For the second part, we may assume (X, D) is strictly log smooth. For each stratum $\Sigma \subset \pi_0(D^{\mathrm{reg}})$, by the integrability assumption, the Hodge bundle is big on a log smooth compactification $\overline{Y}_0$ of a resolution Y_0 of Y_Σ. Thus, there is a closed strict subvariety $Z \subset Y_\Sigma$ containing all subvarieties for which the Hodge bundle is not big on a log smooth compactification of a resolution, namely the image in Y_Σ of the non-big locus of the Hodge bundle in $\overline{Y}_0$. Let $\pi : X' \to X$ be an embedded log resolution of the preimage $f_\Sigma^{-1}(Z)$, and let D' be the union of the (reduced) exceptional divisors and the reduction of $\pi^{-1}(D)$. Then for any stratum $X'_{\Sigma'}$ of (X', D') mapping to X_Σ, the non-big locus of the Hodge bundle of $Y'_{\Sigma'}$ has strictly smaller dimension, and the stratum mapping to $X_\Sigma \backslash f_\Sigma^{-1}(Z)$ satisfies (B1). By induction on the dimensions of the non-big locus in the period image using (1), it follows that for each Σ there is a proper log smooth (X'', D'') and a modification $\sigma : X'' \to X$ with $\sigma^{-1}(D) \subset D''$ for which every stratum mapping to X_Σ satisfies (B1). We may find a proper log smooth (X''', D''') with a modification $\tau : X''' \to X$ with $\tau^{-1}(D) \subset D'''$ which factors through the modification $X'' \to X$ we thereby construct for each Σ, and again using (1) it follows that (X''', D'''') satisfies (B1).

The claim for (B2) follows by the same argument, except in the argument from the previous paragraph, we take $Z \subset Y_\Sigma$ to be the complement of Y_Σ°.

We prove the final claim for $(B3)_R$ by descending induction on the dimension of Y_Σ, the base case being trivial. Thus, assume the condition holds for any $\Sigma \subset \pi_0(D^{\mathrm{reg}})$ with $\dim Y_\Sigma > k$, and consider a stratum Σ with $\dim Y_\Sigma = k$. Suppose that there are strata $X_{\Sigma'}$ such that there are components of $R\,|_{X_\Sigma \times X_{\Sigma'}}$ whose projections are not surjective onto X_Σ. For each such component R_0, let Z_0 be its projection to X_Σ if it is not dominant onto X_Σ, and the complement in X_Σ of its projection if it is dominant. Let $Z \subset X_\Sigma$ be the union of the closures of all of these Z_0s (as R_0 ranges over all components), and now pass to a log resolution of $f_\Sigma^{-1}(\overline{f_\Sigma(Z)})$. By Lemma 3.6, it follows that only strata with period image of dimension strictly smaller than k are produced. On the other hand, the stratum above $X_\Sigma \backslash f_\Sigma^{-1}(\overline{f_\Sigma(Z)})$ now satisfies $(B3)_R$. Continuing in this way, we are done by induction. $\square$

3.4. **The quotient by R_{curve}.** Suppose (X, D) satisfies Property (B1) and for each stratum Σ let $f_\Sigma : X_\Sigma \to Y_\Sigma$ be the period map introduced therein. Define

$$R_\Sigma \coloneqq X_\Sigma \times_{Y_\Sigma} X_\Sigma \subset X_\Sigma \times X_\Sigma$$

and

$$R_{\mathrm{per}} = \bigcup_{\Sigma \subset \pi_0(D^{\mathrm{reg}})} \overline{R_\Sigma}(\mathbb{C}) \subset X(\mathbb{C}) \times X(\mathbb{C})$$

which is a closed, reflexive, symmetric, constructible relation on X. Denote R^e_{per} the equivalence relation it generates.

Lemma 3.8. *Suppose that $V_{\mathbb{Z}}$ has neat monodromy and (X, D) satisfies Property (B1). Then $R_{\mathrm{curve}} = R^e_{\mathrm{per}}$.*

Proof. $R_{\mathrm{curve}} \subset R^e_{\mathrm{per}}$ is immediate from Property 3.5, so it suffices to prove $R_{\mathrm{per}} \subset R_{\mathrm{curve}}$. For any point $(x, y) \in R_{\mathrm{per}}$, there is a smooth curve $g : C \to R_\Sigma$ for some Σ containing (x, y) in the closure of its image. This means we have two maps $C \rightrightarrows X_\Sigma$ with the same composition to Y_Σ. The base-change $(X_\Sigma)_C$ then admits two sections and has geometrically connected generic fiber. There is therefore a surface $S \subset (X_\Sigma)_C$ flat over C containing the two sections whose generic fiber is geometrically connected. There is then a proper surface $\overline{S}$ flat over $\overline{C}$ compactifying S/C with a map $\overline{S} \to X$ extending the map $S \to X$ such that the fibers of $\overline{S}/\overline{C}$ are connected and one of of these fibers F has image in X containing both x and y. The Hodge bundle has degree 0 on F since it does so on the generic fiber of $\overline{S}/\overline{C}$, so $x \sim_{\mathrm{curve}} y$. □

Corollary 3.9. *Let (X, D) be a proper log smooth algebraic space and $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ a polarizable integral pure CY variation on $X\backslash D$ with unipotent local monodromy and integrable Hodge bundle L. Then R_{curve} is a closed constructible equivalence relation on $X(\mathbb{C})$.*

Proof. According to Lemma 3.7 there is a modification $\pi : X' \to X$ such that (X', D') satisfies Property (B1) with respect to the pullback variation. The relation R_{curve} on $X(\mathbb{C})$ is clearly the image of the corresponding relation on $X'(\mathbb{C})$, so the claim follows from Lemma 3.4(4) and Lemma 3.8. □

3.5. **Hodge strata.** Let (X, D) be a proper strictly log smooth algebraic space and $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ a polarizable integral pure CY variation on $X\backslash D$ with unipotent local monodromy and integrable Hodge bundle. Let $R \subset X(\mathbb{C}) \times X(\mathbb{C})$ be a closed algebraic equivalence relation with $R_{\mathrm{curve}} \subset R \subset R_{\mathrm{tr}}$ and assume (X, D) satisfies Property $(\mathrm{B3})_R$. Then for any strata $\Sigma, \Sigma' \subset \pi_0(D^{\mathrm{reg}})$, there exist $x \in X_\Sigma$ and $x' \in X_{\Sigma'}$ such that $x \sim_R x'$ if and only if for every point $x \in X_\Sigma$ there is a point $x' \in X_{\Sigma'}$ such that $x \sim_R x'$. Thus, the saturation of any stratum X_Σ with respect to R is a union of strata.

Definition 3.10. In the above situation, we say $\Sigma \sim_R \Sigma'$ if there are points $x \in X_\Sigma, x' \in X_{\Sigma'}$ with $x \sim_R x'$. We refer to an equivalence class $S \subset P(\pi_0(D^{\mathrm{reg}}))$ with respect to this relation, as well as $X_S := \bigcup_{\Sigma\in S} X_\Sigma$, as an R-stratum. We refer to R_{curve}-strata as Hodge strata.

Lemma 3.11. *In the above situation, suppose (X, D) satisfies Properties (B1-3). Then there exists a local system $V^{\min,\vee}(S)_{\mathbb{Q}} \subset j_*(V^\vee_{\mathbb{Q}}) |_{\mathfrak{T}(S)}$ and a quotient local system $V^{\min,\vee}(S)_{\mathbb{Q}} \to V^{\mathrm{tr},\vee}(S)_{\mathbb{Q}}$ whose restriction to each $\mathfrak{T}(\Sigma)$ for $X_\Sigma \subset X_S$ agrees with $V^{\min,\vee}(\Sigma) \subset j_*(V^\vee_{\mathbb{Q}}) |_{\mathfrak{T}(S)}$ and $V^{\min,\vee}(\Sigma)_{\mathbb{Q}} \to V^{\mathrm{tr},\vee}(\Sigma)_{\mathbb{Q}}$.*

Proof. For any union of strata Z, let $i_Z : Z \to X$ denote the inclusion. By Lemma 2.7, it suffices to show there is a subsheaf $V^{\min,\vee}_{S,\mathbb{Q}} \subset i^*_{X_S} j_*(V^\vee_{\mathbb{Q}})$ (resp. a quotient $V^{\min,\vee}_{S,\mathbb{Q}} \to V^{\mathrm{tr},\vee}_{S,\mathbb{Q}}$) restricting to $V^{\min,\vee}_{\Sigma,\mathbb{Q}} \subset i^*_{X_\Sigma} j_*(V^\vee_{\mathbb{Q}})$ (resp. $V^{\min,\vee}_{\Sigma,\mathbb{Q}} \to V^{\mathrm{tr},\vee}_{\Sigma,\mathbb{Q}}$) for each $X_\Sigma \subset X_S$.

Let $E \cap \overline{X_\Sigma}$ be a boundary component of X_Σ in the same Hodge stratum, which means there is a curve $C \subset X_\Sigma$ whose closure meets $E \cap \overline{X_\Sigma}$ and is contracted by f_Σ. Then by the Griffiths criterion, $V^{\mathrm{tr},\vee}_\Sigma$ and f_Σ extend over $E \cap \overline{X_\Sigma}$, as therefore does $V^{\min,\vee}_\Sigma$. Thus, both extend to the closure $\breve{X}_\Sigma$ of X_Σ in the Hodge stratum X_S containing X_Σ. Call these extensions $V^{\mathrm{tr},\vee}_{\breve{X}_\Sigma}$ and $V^{\min,\vee}_{\breve{X}_\Sigma}$; note that the underlying local system $V^{\min,\vee}_{\breve{X}_\Sigma,\mathbb{Q}}$ is naturally a subsheaf of $i^*_{X_\Sigma} j_*(V^\vee_{\mathbb{Q}})$.

A subsheaf of $i^*_{X_S} j_*(V^\vee_{\mathbb{Q}})$ is uniquely determined by subsheaves of $i^*_Z j_*(V^\vee_{\mathbb{Q}})$ for each closed union of strata $Z \subset X_S$ which agree on intersections. By Property (B3), any stratum $X_{\Sigma'}$ in $\breve{X}_\Sigma$ dominates Y_Σ, so $i^*_{X_{\Sigma'}} V^{\min,\vee}_{\breve{X}_\Sigma,\mathbb{Q}} = V^{\min,\vee}_{X_{\Sigma'},\mathbb{Q}}$, hence there is a subsheaf $V^{\min,\vee}_{S,\mathbb{Q}} \subset i^*_{X_S} j_*(V^\vee_{\mathbb{Q}})$ restricting to each $V^{\min,\vee}_{\Sigma,\mathbb{Q}}$ which is therefore a local system. Likewise, there is a quotient $V^{\min,\vee}_{S,\mathbb{Q}} \to V^{\mathrm{tr},\vee}_{S,\mathbb{Q}}$ restricting to the quotient $V^{\min,\vee}_{\Sigma,\mathbb{Q}} \to V^{\mathrm{tr},\vee}_{\Sigma,\mathbb{Q}}$ for each $X_\Sigma \subset X_S$, and this completes the proof. □

By the previous lemma, we obtain the following diagram by projecting the Hodge bundle to $V^{\min}(S)_{\mathcal{O}}$ which restricts to the diagram in Corollary 2.14 for every $X_\Sigma \subset X_S$, after choosing a path from x_Σ to x_S.

$$\begin{tikzcd}
\widetilde{\mathfrak{T}(S)}^{V^{\mathrm{tr}}(S)_{\mathbb{Q}}} \arrow[rr, "\rho(S)"] & & \mathbb{P}(V^{\min}_{S,\mathbb{C},x(S)})^{\mathrm{an}} =: \mathbb{P}^{\mathrm{an}}_S \\
\widetilde{X_S}^{V^{\mathrm{tr}}_{S,\mathbb{Q}}} \arrow[u] \arrow[r, "\phi^{\mathrm{tr}}_S"] \arrow[rr, bend right, "\rho_S"'] & \check{\mathbb{D}}(V^{\mathrm{tr}}_{S,\mathbb{C},x_S})^{\mathrm{an}} \arrow[r] & \mathbb{P}(V^{\mathrm{tr}}_{S,\mathbb{C},x_S})^{\mathrm{an}}. \arrow[u]
\end{tikzcd} \tag{3.1}$$

Proposition 3.12. *Let (X, D) be a proper log smooth algebraic space and $V = (V_{\mathbb{Z}}, F^\bullet V_{\mathcal{O}})$ a polarizable integral pure CY variation on $X \backslash D$ with neat monodromy, integrable Hodge bundle L, and (X, D) satisfying Property (B1-3). For any Hodge stratum S, the morphism $\rho(S)$ from (3.1) is π_1-definable analytic as in Definition 2.6. The pullback of the Hodge bundle to $\widetilde{\mathfrak{T}(S)}^{V^{\mathrm{tr}}(S)_{\mathbb{Q}}}$ is naturally identified (as a π_1-definable analytic line bundle) with the pullback of $\mathcal{O}_{\mathbb{P}^{\mathrm{def}}_S}(1)$. Finally, if the Hodge bundle has torsion combinatorial monodromy, then the connected components of the fibers of ρ_S are identified via the covering map with the fibers of $X_S \to \mathfrak{Y} := X(\mathbb{C})/R_{\mathrm{curve}}$. In particular, they are compact.*

Proof. The definability is clear by [BM23], as is the statement about the Hodge bundle, so it remains to prove the statement about the fibers. Let $\widetilde{F}$ be a fiber of ρ_S. Since $\tilde{F}$ is a definable closed subspace which is π_1-stable, by definable Chow it is the inverse image of a closed algebraic $F \subset X_S$. The Hodge bundle is flat along each component of F, and since the boundary satisfies Property (B1), the variation $V^{\mathrm{tr}}_{S,\mathbb{Q}}$ is isotrivial on F. For any irreducible curve C in F and $\overline{C}$ the closure in X, the local monodromy of $V^{\mathrm{tr}}_{S,\mathbb{Q}}|_C$ is then trivial, so $\overline{C} \subset X_S$ and therefore $\overline{C} \subset F$. Thus, F is proper. Clearly F maps to a point in $\mathfrak{Y}$; for any proper connected curve C in a fiber of $|X| \to \mathfrak{Y}$ meeting F, the inverse image in $\widetilde{X_S}^{V^{\mathrm{tr}}_{S,\mathbb{Q}}}$ is contained in a fiber of ρ_S and meets $\tilde{F}$, hence is contained in $\widetilde{F}$. Thus, $C \subset F$, and F is a full fiber of $|X| \to \mathfrak{Y}$. Finally, by Lemma 3.3 and the Lemma 3.13 below, the monodromy of $V^{\mathrm{tr}}_{S,\mathbb{Q}}|_F$ is finite since the Hodge bundle has torsion combinatorial monodromy, hence trivial by neatness. It follows that every connected component of $\widetilde{F}$ is a copy of F. □

Lemma 3.13. *Let F be a connected algebraic space. Then there is a nodal curve $g : C \to F$ such that $g_* : \pi_1(C^{\mathrm{an}}) \to \pi_1(F^{\mathrm{an}})$ is surjective.*

Proof. Note that an algebraic space is locally contractible, and so it is enough to show that for any point $p \in F$ there exists a contractible neighborhood $p \in U \subset F$ that the points of U are locally connected via algebraic curves. To prove this, first note that F locally has an étale cover by a scheme over p, so it is enough to consider the case of F a scheme. Now, in this case, replacing F with some affine subscheme, we may assume that F has a finite surjective map to an open subscheme of affine space, and since the pullback of a curve under a finite map is still a curve, the claim follows. □

3.6. **Algebraizations.**

Definition 3.14. Let X be an algebraic space, and $\phi : |X| \to \mathfrak{M}$ be a continuous surjective map of definable topological spaces. We say ϕ is algebraic with source X (or just algebraic if X is clear from context) if there exists a morphism of algebraic spaces $f : X \to Y$ and an identification $\mathfrak{M} \cong |Y|$ such that $|f| : |X| \to |Y|$ is identified with ϕ via this identification.

Note that the algebraic space Y is not unique, but if ϕ is proper with connected fibers, we may require $\mathcal{O}_Y \to f_*\mathcal{O}_X$ to be an isomorphism, in which case the algebraic space structure on Y is unique.

Lemma 3.15. *Let (X, D) be a proper log smooth algebraic space and $V = (V_\mathbb{Z}, F^\bullet V_\mathcal{O})$ a polarizable integral pure CY variation on $X\backslash D$ with neat monodromy, integrable Hodge bundle L with torsion combinatorial monodromy, and (X, D) satisfying Property (B1-3). Let $q\colon |X| \to \mathfrak{Y} \coloneqq X(\mathbb{C})/R_{\mathrm{curve}}$ be the quotient map, $X_\mathfrak{U} \subset X$ an open union of Hodge strata, and $\mathfrak{U} \coloneqq q(|X_\mathfrak{U}|)$ the image. If $|X_\mathfrak{U}| \to \mathfrak{U}$ is algebraized by a fibration $f_\mathfrak{U} : X_\mathfrak{U} \to Y_\mathfrak{U}$, then for some $k > 0$, $L^k|_{X_\mathfrak{U}} = f_\mathfrak{U}^* A$ for an ample line bundle A on $Y_\mathfrak{U}$. Moreover, the vanishing sections (in the sense of Theorem 2.5) of some power of A define a locally closed embedding $Y_\mathfrak{U} \to \mathbb{P}^N$.*

Proof. By Proposition 3.12 the line bundle L is trivial on the completion of X along any fiber, hence descends to L_Y on $Y_\mathfrak{U}$. For each Hodge stratum $X_S \subset X_\mathfrak{U}$, the image $Y_S \subset Y_\mathfrak{U}$ is algebraic, and since $X_S \to Y_S$ has connected fibers, the finite part $g : Y'_S \to Y_S$ of the Stein factorization $X_S \to Y'_S \to Y_S$ is a homeomorphism, and in particular, birational. By Lemma 2.10 the variation V_S^{tr} descends to Y'_S and has $g^* L_{Y_S}$ as its Hodge bundle. By Lemma 2.16 the Hodge bundle satisfies the requirements of Setup 2.3, so the claim then follows from Theorem 2.5. □

4. Semiampleness

In this section, we prove the following:

Theorem 4.1. *Let (X, D) be a proper log smooth algebraic space, $(V_\mathbb{Z}, F^\bullet V_\mathcal{O})$ a polarizable integral pure CY variation of Hodge structures on $X\backslash D$ with unipotent local monodromy, and L the Hodge bundle on X. If L is integrable with torsion combinatorial monodromy, then it is semiample.*

We deduce the same statement for the Griffiths bundle, where the integrability and combinatorial monodromy conditions are automatic:

Corollary 4.2. *Let (X, D) be a proper log smooth algebraic space, $({}_{\mathrm{orig}}V_\mathbb{Z}, F^\bullet{}_{\mathrm{orig}}V_\mathcal{O})$ a polarizable integral pure variation of Hodge structures on $X\backslash D$ with unipotent local monodromy, and L the Griffiths bundle on X. Then L is semiample.*

Proof of Corollary 4.2 given Theorem 4.1. Apply Lemma 2.19 and Theorem 2.22. □

Remark 4.3. We remark that both the integrability of the Hodge bundle and the torsion combinatorial monodromy condition can clearly be checked after pulling back along a dominant proper morphism $(X', D') \to (X, D)$.

4.1. Proof of Theorem 4.1.

The main step in the proof of Theorem 4.1 is the following:

Theorem 4.4. *Let (X, D) and $V = (V_\mathbb{Z}, F^\bullet V_\mathcal{O})$ be as in Theorem 4.1. Then the quotient $q : |X| \to \mathfrak{Y} \coloneqq X(\mathbb{C})/R_{\mathrm{curve}}$ is algebraic.*

Proof. We first reduce to the case where the monodromy of $V_\mathbb{Z}$ is neat. Let L be the Hodge bundle of X. By adjoining enough level structure, there is a proper log smooth algebraic space (X', D') and a morphism $g : X' \to X$ restricting to a finite étale cover $g|_{X'\backslash D'} : X'\backslash D' \to X\backslash D$ such that $g|^*_{X'\backslash D'} V_\mathbb{Z}$ has neat monodromy and $X\backslash D$ is the quotient of $X'\backslash D'$ be a finite fixed-point free group action by G. The finite part $h : X'' \to X$ of the Stein factorization of $X' \to X$ is then the normalization of X in the function field of X', so the group action by G extends to X'' and X is the quotient. Thus, there is a norm map from sections of $h^* L$ to sections of $L^{|G|}$, so L is semiample if and only if $h^* L$ is, which in turn is semiample if and only if $g^* L$ is, since $X' \to X''$ is a fibration. Thus, we may assume the monodromy is neat.

By Lemma 3.7 we may assume (X, D) satisfies Property (B1-3) after replacing (X, D) with a modification. Each Hodge stratum X_S is saturated with respect to the quotient map q, and we therefore obtain a locally closed (in the quotient Zariski topology) stratification $\mathfrak{Y}_S \coloneqq q(|X_S|)$ of $\mathfrak{Y}$.

Claim 4.5. *Let $\mathfrak{U} \subset \mathfrak{Y}$ be an open union of strata, and $X_{\mathfrak{U}} \subset X$ the open subspace with underlying topological space $q^{-1}(\mathfrak{U})$. Then $|X_{\mathfrak{U}}| \to \mathfrak{U}$ is algebraic.*

We prove the claim by induction, adding one stratum at a time. The base case (the case of the open stratum) is a consequence of [BBT23a, Theorem 1.1]. For the general case, we may assume there is a stratum $\mathfrak{Y}_S \subset \mathfrak{U}$ which is closed in $\mathfrak{U}$ and such that, setting $\mathfrak{U}' = \mathfrak{U} \backslash \mathfrak{Y}_S$, $|X_{\mathfrak{U}'}| \to \mathfrak{U}'$ is algebraized by a fibration $f_{\mathfrak{U}'} : X_{\mathfrak{U}'} \to U'$. According to Lemma 3.15, there is a finite-dimensional space of sections of L^k on X which yields a morphism $X_{\mathfrak{U}'} \to \mathbb{P}^{N'}$ which factors as $X_{\mathfrak{U}'} \to U' \to \mathbb{P}^{N'}$ where $U' \to \mathbb{P}^{N'}$ is a locally closed embedding.

Recall that by Corollary 3.9 the quotient $\mathfrak{U} = X_{\mathfrak{U}}(\mathbb{C})/R_{\text{curve}}$ naturally exists in the category of definable topological spaces [vdD98, Chap. 10, (2.15) Theorem]. Take a DR-neighborhoods $X_S \subset \mathfrak{T}(S) \subset X_{\mathfrak{U}}$, $\mathfrak{Y}_S \subset \mathfrak{T}_{\mathfrak{Y}}(S) \subset \mathfrak{U}$ for which $X_{\mathfrak{T}_{\mathfrak{Y}}(S)} := q^{-1}(\mathfrak{T}_{\mathfrak{Y}}(S)) \subset \mathfrak{T}(S)$. By Proposition 3.12, we obtain a π_1-definable analytic morphism $\rho(S) : \widetilde{\mathfrak{T}(S)}^{V^{\mathrm{tr}}(S)_{\mathbb{Q}}} \to \mathbb{P}^{\mathrm{an}}_S$. Moreover, the restriction of $V^{\mathrm{tr}}(S)_{\mathbb{Q}}$ descends to $V^{\mathrm{tr}}_{\mathfrak{Y}}(S)_{\mathbb{Q}}$ on $\mathfrak{T}_{\mathfrak{Y}}(S)$ and the restriction of $\rho(S)$ to $\widetilde{X_S}^{V^{\mathrm{tr}}_{S,\mathbb{Q}}}$ factors through $\widetilde{\mathfrak{Y}_S}^{V^{\mathrm{tr}}_{\mathfrak{Y},S,\mathbb{Q}}}$. Taking the pullback of the linear system $|\mathcal{O}_{\mathbb{P}_S}(k)|$ and combining it with the previous linear system of sections of L^k we obtain a π_1-definable analytic morphism

$$\sigma(S) : \widetilde{X_{\mathfrak{T}_{\mathfrak{Y}}(S)}}^{V^{\mathrm{tr}}(S)_{\mathbb{Q}}} \to (\mathbb{P}^N)^{\mathrm{an}}$$

has fibers with compact connected components and the finite part of whose Stein factorization is locally a closed immersion away from $\widetilde{X_S}^{V^{\mathrm{tr}}_{S,\mathbb{Q}}}$. In fact, the connected components of the fibers in $\widetilde{X_S}^{V^{\mathrm{tr}}_{S,\mathbb{Q}}}$ are identified with those of the quotient $|X_{\mathfrak{U}}| \to \mathfrak{U}$ via the covering map since they are contained in the latter by the above, and every Hodge degree 0 curve in X_S lifts to $\widetilde{X_S}^{V^{\mathrm{tr}}_{S,\mathbb{Q}}}$ and must be contracted. Thus, the *topological* Stein factorization of $\sigma(S)$ is $\widetilde{\mathfrak{T}_{\mathfrak{Y}}(S)}^{V^{\mathrm{tr}}_{\mathfrak{Y}}(S)_{\mathbb{Q}}}$.

As in [BBT24], by definable triangulation, there is a definable cover $\mathfrak{U}_i$ of $\mathfrak{T}_{\mathfrak{Y}}(S)$ by contractible open subsets which therefore lift to $\widetilde{\mathfrak{T}_{\mathfrak{Y}}(S)}^{V^{\mathrm{tr}}_{\mathfrak{Y}}(S)_{\mathbb{Q}}}$. Letting $X_{\mathfrak{U}_i} := q^{-1}(\mathfrak{U}_i) \subset X^{\mathrm{def}}$ be the corresponding open definable analytic subspaces which lift to $\widetilde{X_{\mathfrak{T}_{\mathfrak{Y}}(S)}}^{V^{\mathrm{tr}}(S)_{\mathbb{Q}}}$, by definable Stein factorization [BBT24, Theorem 1.7] there is a Stein factorization $X_{\mathfrak{U}_i} \to \mathcal{U}_i \to (\mathbb{P}^N)^{\mathrm{def}}$ in the category of definable analytic spaces, and $|\mathcal{U}_i|$ is canonically identified with $\mathfrak{U}_i$ via the quotient map, by the above. We therefore obtain a definable analytic space structure on $\mathfrak{T}_{\mathfrak{Y}}(S)$ for which the quotient map $|X_{\mathfrak{T}_{\mathfrak{Y}}(S)}| \to \mathfrak{T}_{\mathfrak{Y}}(S)$ underlies a morphism of definable analytic spaces and is compatible with $f^{\mathrm{def}}_{\mathfrak{U}'} : X^{\mathrm{def}}_{\mathfrak{U}'} \to U'^{\mathrm{def}}$, hence we obtain a morphism of definable analytic spaces $X^{\mathrm{def}}_{\mathfrak{U}} \to \mathcal{U}$ which is identified with $|X_U| \to \mathfrak{U}$ on the level of definable topological spaces. By the definable image theorem (Theorem 2.2), this morphism is algebraic, whence the claim. □

Proof of Theorem 4.1. Combine Lemma 3.15 and Theorem 4.4. □

4.2. **Examples and counterexamples.** The integrability and combinatorial monodromy conditions in Theorem 4.1 are clearly both necessary. We give some examples showing that neither condition is sufficient on its own.

Example 4.6. We give an example of a CY variation whose Hodge bundle is integrable but has nontorsion combinatorial monodromy:

Let $F/\mathbb{Q}$ be a real quadratic field with ring of integers $\mathcal{O}_F$. Consider the lattice $U_{\mathbb{Z},0} := \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}\, \mathcal{O}_F^2$ with $q_0 := \mathrm{tr}_{F/\mathbb{Q}}\langle\, ,\, \rangle$ where $\langle\, ,\, \rangle$ is the standard $\mathcal{O}_F$-linear antisymmetric pairing on $\mathcal{O}_F^2$. The two embeddings $\iota_1, \iota_2 : F \to \mathbb{R}$ give embeddings $\mathrm{SL}_2(\mathcal{O}_F) \to \mathrm{SL}_2(\mathbb{R})$, and the Hilbert modular surface $X = \mathrm{SL}_2(\mathcal{O}_F)\backslash\mathbb{H}^2$ parametrizes Hodge structures on $U_{\mathbb{Z},0}$ polarized by q_0 for which the splitting over $\mathbb{R}$ into ι_1 and ι_2 eigenspaces is a decomposition of Hodge structures. Let $U = (U_{\mathbb{Z}}, F^\bullet U_{\mathcal{O}}, q)$ be the associated polarized variation of Hodge structures on X. Then $\wedge^2 U$ is a CY variation whose Hodge bundle is the Griffiths bundle of U, hence semiample.

Equip $\mathbb{Z}(0)^2$ with the standard diagonal polarization λ and consider the polarizable integral variation of Hodge structures

$$V = \mathbb{Z}(0)^2 \otimes_{\mathbb{Z}(0)} U$$

on X which is polarized by $\lambda \otimes q$. Let $e := e_1 + ie_2 \in \mathbb{C}(0)^2$, which is λ-isotropic, and let $U_{\mathbb{R}} = U_1 \oplus U_2$ be the splitting over $\mathbb{R}$ into the two eigenspaces. Consider the variation V' which has the same underlying integral local system as V, but the Hodge filtration is changed by shifting:

$$V'_{\mathcal{O}} = \mathbb{C}e \otimes_{\mathbb{C}} (U_1)_{\mathcal{O}}(2,-2) \oplus \mathbb{C}(0)^2 \otimes_{\mathbb{C}} (U_2)_{\mathcal{O}} \oplus \mathbb{C}\overline{e} \otimes_{\mathbb{C}} (U_1)_{\mathcal{O}}(-2,2).$$

Since $\mathbb{C}e \otimes_{\mathbb{C}} U_1$ is a $\lambda \otimes q$-isotropic sub-$\mathbb{C}$-variation and its conjugate is $\mathbb{C}\overline{e} \otimes_{\mathbb{C}} U_1$, the shift defines a new $\mathbb{C}$-variation and is still polarized by $\lambda \otimes q$. It has the following properties:

(1) V' is a polarizable integral pure CY variation with Hodge bundle $F^3 V'_{\mathcal{O}} = F^1(U_1)_{\mathcal{O}}$. Thus, its Hodge bundle is trivial along the leaves of one of the two transcendental foliations of X given by the product structure on $\mathbb{H}^2$, and so the Hodge bundle is not integrable.

(2) After passing to a finite-index $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$, X has a log smooth compactification, the connected components of whose boundary are cycles of rational curves. The Hodge bundle $F^2\mathcal{V}'$ is trivial on each of these curves. The monodromy of $U_{\mathbb{Z}}$ in a neighborhood of one of those connected components is given by upper-triangular matrices (see e.g. [AMRT10, §I.5])

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}$$

for α ranging over a finite index subgroup of units in $\mathcal{O}_F$ and β ranging over some ideal class of $\mathcal{O}_F$. The combinatorial monodromy on $F^1\mathcal{U}_j$ around the cycle is given by multiplication by $\iota_j(\alpha^2)$, and so the Hodge bundle of V' does not have torsion combinatorial monodromy (nor does it have norm one combinatorial monodromy, in accordance with Lemma 2.23).

The CY variation $V' \otimes \wedge^2 U$ then has integrable Hodge bundle, but nontorsion combinatorial monodromy.

Note also that U_1 from Example 4.6 shows that the integral structure is important in the statement of Corollary 4.2, and that a $\overline{\mathbb{Z}}$-structure is not sufficient.

Example 4.7. We give an example of a CY variation whose Hodge bundle has torsion combinatorial monodromy (because there is no boundary) but is non-integrable:

We have the following variation of the above example: fixing a degree 4 CM field $K/\mathbb{Q}$, $\langle\, ,\, \rangle$ the standard hermitian form on $\mathcal{O}_K^2$, and taking $V_{\mathbb{Z},0} := \mathrm{Res}_{\mathcal{O}_K/\mathbb{Z}}\, \mathcal{O}_K^2$ with $q_0 := \mathrm{tr}_{K/\mathbb{Q}}\langle\, ,\, \rangle$, X parametrizes Hodge structures on $V_{\mathbb{Z},0}$ polarized by q_0 such that the eigenspaces associated to each embedding $K \to \mathbb{C}$ are complex sub-Hodge structures. The variation V' is obtained from the resulting variation V by shifting a conjugate pair of factors in this decomposition.

Fix a degree $2n \geq 6$ CM field $K/\mathbb{Q}$, and a hermitian pairing $\langle\, ,\, \rangle$ on $\mathcal{O}_K^2$ which is indefinite in exactly 2 places. Performing the same shifting construction, we obtain a CY variation on a variety X with no boundary and whose Hodge bundle is not integrable. Moreover, the Hodge bundle in this case is strictly nef since there are no curves which lift to a fiber of $\mathbb{H} \times \mathbb{H}$ and thus the Hodge bundle vacuously has torsion combinatorial monodromy.

Remark 4.8. Neither of the variations in Example 4.6 and Example 4.7 is manifestly algebraic, and it is not clear to the authors whether a geometric polarizable integral pure CY variation automatically has Hodge bundle which is integrable and has torsion combinatorial monodromy. Of course, Theorem 7.1 and

Theorem 7.2 below show this is the case for the variation on middle cohomology for a family of K-trivial varieties.

Example 4.9. In both Theorem 4.1 and Corollary 4.2, the assumption of unipotent local monodromy is necessary. This can even be seen on the level of local systems with finite monodromy. Let $L = \mathcal{O}_{\mathbb{P}^1}(d)$ and consider the ruled surface $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus L)$, which is the total spaces of L and $L^\vee$ glued along the canonical map $L\backslash 0 \to L^\vee\backslash 0 : s \mapsto s^\vee$. Let s_0, s_∞ be the 0 sections of $L, L^\vee$, respectively, thought of as sections of $X \to \mathbb{P}^1$. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on X by scaling by ± 1, let $\pi : X \to Y = G\backslash X$ be the quotient, and note that (Y, D) with $D := \pi(s_0) + \pi(s_\infty)$ is log smooth. Let $V_\mathbb{Z} = (\pi_*\mathbb{Z}_{(X\backslash D)^{\mathrm{an}}})^-$ be the local system on $(Y\backslash D)^{\mathrm{an}}$ of anti-invariant locally constant functions on $(X\backslash s_0 \cup s_\infty)^{\mathrm{an}}$. The local monodromy around each component of D is ± 1, and the Deligne extension $\mathcal{V}$ is generated by $q^{-1/2}v$ where v is a flat section and q is a local defining equation. The pullback π^*V has a global flat section which extends with simple zeroes along s_0 and s_∞ as a section of $\pi^*\mathcal{V}$, so we have $\pi^*\mathcal{V} \cong \mathcal{O}_X(s_0 + s_\infty)$. In particular, $\pi^*\mathcal{V}|_{s_0} \cong L$ and $\pi^*\mathcal{V}|_{s_\infty} \cong L^\vee$, so for $d \neq 0$, $\mathcal{V}$ is not semiample.

Remark 4.10. There is however a natural modification of a power of the Griffiths bundle which is semiample when the local monodromy is not unipotent. Indeed, for any polarizable integral pure variation of Hodge structures $V = (V_\mathbb{Z}, F^\bullet V_\mathcal{O})$ on a log smooth algebraic space (X, D), by adjoining enough level structure there is a finite cover $\pi : X' \to X$ with X' normal such that X is the quotient of X' by a group action G and such that $V' := \pi^*_{X\backslash D}V$ has unipotent local monodromy, where $\pi_{X\backslash D}$ is the corestriction to $X\backslash D$. Taking a log resolution $\pi' : X'' \to X$, a power L'^k of the Griffiths bundle L' of $V'' := \pi'^*_{X'\backslash D'}V'$ descends to a line bundle $L^{(k)}$ on X, and using the norm map we deduce that $L^{(k)}$ is semiample. Likewise for the Hodge bundle, assuming the integrability and combinatorial monodromy conditions (on X). This will be the natural polarization of the Baily–Borel discussed in the next section.

5. Baily–Borel compactifications of period images

5.1. **Preliminaries.** Let X be a smooth algebraic space and $\phi : X^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ be a period map associated to a polarizable integral pure variation of Hodge structures, and Γ is any discrete group containing the image of the monodromy representation. By [BBT23a, Theorem 1.1] (see also [BBT23b, Corollary 2.11]) the closure of the image is naturally a quasiprojective variety: there is a unique dominant morphism $f : X \to Y$ to a quasiprojective variety Y and a closed immersion $\iota : Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ such that $\phi = \iota \circ f^{\mathrm{an}}$. Our goal in this section is to prove the existence of a canonical compactification Y^{BB} of Y which we call the *Baily–Borel* compactification.

The following is a more intrinsic and slightly more general notion than image of a period map. Let Y be a reduced and irreducible algebraic space with a quasifinite Griffiths transverse[8] morphism $\phi : Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$, where Γ is a discrete group preserving an integral lattice. After passing to a finite morphism $f : Y' \to Y$ by adjoining level structure, there will be a polarizable integral pure variation of Hodge structure ${}_{\mathrm{orig}}V'$ on Y' with monodromy contained in Γ, and the period map $Y' \to \Gamma\backslash\mathbb{D}$ will factor through ϕ. In this situation, it follows from [BBT23a] that the Griffiths bundle L_Y is ample on Y. Henceforth, we refer to such a Y as a "variety with quasifinite period map."

Now, for general Γ the Griffiths bundle exists on the stack $[\Gamma\backslash\mathbb{D}]$ and a power of it $L^{k_\Gamma}_{[\Gamma\backslash\mathbb{D}]}$ descends to the coarse space $\Gamma\backslash\mathbb{D}$. We shall use the notation $L^{(k_\Gamma)}_{\Gamma\backslash\mathbb{D}}$ for the descent, as the descent is not necessarily a power of a line bundle.

Definition 5.1. Let Y be a reduced and irreducible algebraic space with a Griffiths transverse morphism $\phi : Y \to \Gamma\backslash\mathbb{D}$. For each non-negative integer n we define $H^0_{\mathrm{mg}}(Y, L_Y^{(nk_\Gamma)}) \subset H^0(Y, L_Y^{(nk_\Gamma)})$ to be those sections

[8]That is, for any resolution $Z \to Y$, $Z^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ is Griffiths transverse.

s whose Hodge norm $|s|$ grows sub-polynomially at the boundary. We call these the sections *of moderate growth*. We define the associated graded ring $B_Y := \bigoplus_{n\geq 0} H^0_{\mathrm{mg}}(Y, L_Y^{(nk_\Gamma)})$ with $H^0_{\mathrm{mg}}(Y, L_Y^{(nk_\Gamma)})$ given degree nk_Γ.

Note that for any proper log smooth algebraic space (X, D) and dominant morphism $\pi : X\backslash D \to Y$ for which the composition with the period map $\phi \circ \pi^{\mathrm{an}}$ is locally liftable (to $\mathbb{D}$) and such that the induced local system has unipotent local monodromy, a section $s \in H^0(Y, L_Y^{(nk_\Gamma)})$ has Hodge norm of moderate growth if and only if its pullback $\pi^* s$ extends to a section of the Schmid extension $(L^{nk_\Gamma}_{X\backslash D})_X$ of the nk_Γ-th power of the Griffiths bundle $L_{X\backslash D}$ of the pullback variation ${}_{\mathrm{orig}}V_{X\backslash D}$ by [Kas85]. Note also that for any morphism $f : (X', D') \to (X, D)$ of proper log smooth algebraic spaces and any polarizable integral pure variation of Hodge structures ${}_{\mathrm{orig}}V$ on $X\backslash D$, there is always an injection $f^*(L^k_{X\backslash D})_X \to (L^k_{X\backslash D})_X$ of Schmid extensions of powers of the associated Griffiths bundles. Thus, for any morphism $g : Z \to Y$, we get an induced pullback map $g^* : B_Y \to B_Z$.

Our main result is then as follows:

Theorem 5.2. *Let Y be a variety with quasifinite period map. Then*

1. *B_Y is finitely generated, $Y^{\mathrm{BB}} := \operatorname{Proj} B_Y$ is a projective variety, and the natural morphism $j : Y \hookrightarrow Y^{\mathrm{BB}}$ is an open embedding.*
2. *There exists a minimal positive $k_\Gamma | k_Y$ such that locally on Y^{BB} there are sections of $L_Y^{(k_Y)}$ whose Hodge norms and inverse Hodge norms have moderate growth.*
3. *For $k_Y | n$, $\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ exists as a line bundle and is ample for n positive. The natural inclusion $\mathcal{O}_{Y^{\mathrm{BB}}}(n) \subset j_*(L_Y^{(n)})$ is the subsheaf of sections of moderate growth.*
4. *Let (Z, D_Z) be a log smooth algebraic space and $g : Z\backslash D_Z \to Y$ a morphism for which the composition $(Z\backslash D_Z)^{\mathrm{an}} \xrightarrow{g^{\mathrm{an}}} Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ is locally liftable. Then $g : Z\backslash D_Z \to Y$ extends to a morphism $\overline{g} : Z \to Y^{\mathrm{BB}}$ and for $k_Y | n$, $\overline{g}^*\mathcal{O}_{Y^{\mathrm{BB}}}(n)$ is canonically identified with the Schmid extension $(L^n_{Z\backslash D_Z})_Z$ of the nth power of the Griffiths bundle $L_{Z\backslash D_Z}$ of the induced variation ${}_{\mathrm{orig}}V_{Z\backslash D_Z}$ on $Z\backslash D_Z$.*

We remark that the above properties generalize the construction for Shimura varieties, which is why we call it the Baily–Borel compactification.

5.1.1. *Hodge case.* If Y is a variety with quasifinite period map $\phi : Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ and $\mathbb{D}$ parametrizes CY Hodge structures, we may instead consider the Hodge bundle $M_Y^{(\ell_\Gamma)}$ and likewise define the moderate growth sections $H^0_{\mathrm{mg}}(Y, M_Y^{(n\ell_\Gamma)}) \subset H^0(Y, M_Y^{(n\ell_\Gamma)})$ and $C_Y := \bigoplus_{n\geq 0} H^0_{\mathrm{mg}}(Y, M_Y^{(n\ell_\Gamma)})$ with $H^0_{\mathrm{mg}}(Y, M_Y^{(n\ell_\Gamma)})$ given degree $n\ell_\Gamma$. Note that in this case $M_Y^{(\ell_\Gamma)}$ may not be ample.

We say that:

- $M_Y^{(\ell_\Gamma)}$ is strictly nef if for any nonconstant irreducible smooth curve $g : C \to Y$ for which the composition $C^{\mathrm{an}} \xrightarrow{g^{\mathrm{an}}} Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ is locally liftable and the resulting variation V_C has unipotent local monodromy, the Schmid extension $M_{\overline{C}}$ of the Hodge bundle M_C to the smooth compactification $C \subset \overline{C}$ has positive degree.
- $M_Y^{(\ell_\Gamma)}$ is integrable (resp. has torsion combinatorial monodromy) if for some (hence any) proper log smooth algebraic space (Z, D_Z) with a proper dominant generically finite morphism $g : Z\backslash D_Z \to Y$ for which the composition $(Z\backslash D_Z)^{\mathrm{an}} \xrightarrow{g^{\mathrm{an}}} Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ is locally liftable and the resulting variation $V_{Z\backslash D_Z}$ has unipotent local monodromy, the Hodge bundle M_Z of $V_{Z\backslash D_Z}$ is integrable (resp. has torsion combinatorial monodromy).

Theorem 5.3. *Let Y be a normal variety with quasifinite period map to a period space parametrizing CY Hodge structures. Assume the Hodge bundle $M_Y^{(\ell_\Gamma)}$ is strictly nef, integrable, and has torsion combinatorial monodromy in the above sense. Then*

(1) C_Y is finitely generated, $Y^{\mathrm{BBH}} := \operatorname{Proj} C_Y$ is a normal projective variety, and the natural morphism $j : Y \hookrightarrow Y^{\mathrm{BBH}}$ is an open embedding.

(2) There exists a minimal positive $\ell_\Gamma | \ell_Y$ such that locally on Y^{BBH} there are sections of $M_Y^{(\ell_Y)}$ whose Hodge norms and inverse Hodge norms have moderate growth.

(3) For $\ell_Y | n$, $\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ exists as a line bundle and is ample for n positive. The natural inclusion $\mathcal{O}_{Y^{\mathrm{BBH}}}(n) \subset j_*(M_Y^{(n)})$ is the subsheaf of sections of moderate growth.

(4) Let (Z, D_Z) be a log smooth algebraic space and $g : Z \backslash D_Z \to Y$ a morphism for which the composition $(Z\backslash D_Z)^{\mathrm{an}} \xrightarrow{g^{\mathrm{an}}} Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ is locally liftable. Then $g : Z\backslash D_Z \to Y$ extends to a morphism $\overline{g} : Z \to Y^{\mathrm{BBH}}$ and for $\ell_Y | n$, $\overline{g}^*\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ is canonically identified with the Schmid extension $(M^n_{Z\backslash D_Z})_Z$ of the nth power of the Hodge bundle $M_{Z\backslash D_Z}$ of the induced variation $V_{Z\backslash D_Z}$ on $Z\backslash D_Z$.

Moreover, Y^{BBH} is the unique normal compactification of Y for which a sufficiently divisible power $M_Y^{(n)}$ extends to an ample line bundle and the above property is satisfied.

Remark 5.4. Assuming the hypotheses of Theorem 5.3, there is a natural morphism $Y^{\mathrm{BB}} \to Y^{\mathrm{BBH}}$. This morphism often has positive-dimensional fibers, even on the part of Y^{BB} which maps to $\Gamma\backslash\mathbb{D}$—see Section 7.3.2 for an example coming from a moduli space of Calabi–Yau varieties.

5.1.2. *Borel extension.* We finally prove that the compactifications Y^{BB} and Y^{BBH} satisfy an extension theorem just like in the classical cases:

Theorem 5.5. *Let Y be a variety with quasifinite period map (resp. a variety with quasifinite CY period map satisfying the hypotheses Theorem 5.3). Then any analytic morphism from a polydisk $\phi : (\Delta^*)^k \to Y^{\mathrm{an}}$ such that the resulting morphism $(\Delta^*)^k \to \Gamma\backslash\mathbb{D}$ is locally liftable extends to a morphism $\overline{\phi} : \Delta^k \to Y^{\mathrm{BB,an}}$ (resp. $\overline{\phi} : \Delta^k \to Y^{\mathrm{BBH,an}}$). Moreover, $\overline{\phi}^*\mathcal{O}_{Y^{\mathrm{BB}}}(n)^{\mathrm{an}}$ (resp. $\overline{\phi}^*\mathcal{O}_{Y^{\mathrm{BBH}}}(n)^{\mathrm{an}}$) is canonically identified with the Schmid extension of $L^n_{(\Delta^*)^k}$ (resp. $M^n_{(\Delta^*)^k}$) for $k_Y | n$ (resp. $\ell_Y | n$).*

The rest of the section is devoted to the proofs of Theorem 5.2, Theorem 5.3, and Theorem 5.5. We begin with the following compatibility lemma:

Lemma 5.6. *Let (X, D) be a log smooth algebraic space and V a polarizable integral pure variation of Hodge structures on $X\backslash D$ with unipotent local monodromy. Let $f : (X', D') \to (X, D)$ be a morphism of log smooth algebraic spaces. Then letting $L_X, L_{X'}$ be the Schmid extensions of the Griffiths bundle, we have $f^*L_X = L_{X'}$. Moreover, if V is a CY-variation, the same compatibility holds for the Hodge bundles.*

Proof. The Lemma follows from the fact that the Deligne extension on (X, D) pulls back to the Deligne extension on (X', D'). Recall that the Deligne extension as the unique extension with residues having eigenvalues with real part in $(-1, 0]$. In the case of unipotent monodromy, the eigenvalues are 0, and so the same follows for the pullback. □

5.2. **Reduction to the neat case.** We first reduce parts (1), (2), and (3) of Theorem 5.2 and Theorem 5.3 to the case when the variation in question has neat monodromy. The argument is the same in both cases, so we do it for Theorem 5.2. By adjoining enough level structure, there is a finite morphism $\pi : Y' \to Y$ which is the quotient map for a finite group action G and such that the induced map $\phi' : Y' \to \Gamma\backslash\mathbb{D}$ is the period map of a variation with neat monodromy. In both cases, the pullback $\pi^* : B_Y \to B_{Y'}$ is the inclusion of the G-invariant subring, and by the existence of the global norm map it follows that if (1) holds for Y', then it also holds for Y by taking the quotient of Y'^{BB} by G. The existence of the local norm map implies

(2) locally, so the set of k_Y in question is nonempty. The set of all integral k_Y satisfying (2) clearly form a (nontrivial) group, so taking the minimal positive one, (2) follows. Given (2), part (3) will now follow if the sheaf of moderate growth sections of $j_*(L_Y^{(n)})$ is a line bundle, but since this is the G-invariants of the corresponding subsheaf on Y'^{BB}, this is clear.

5.3. **Proof of Theorem 5.3 (1), (2), and (3).** By the above reduction, we may assume Γ is neat. By resolution of singularities, let (X, D) be a proper log smooth algebraic space with a proper birational morphism $X\backslash D \to Y$. Since Y is normal, it follows that $X\backslash D \to Y$ is a fibration. Applying Theorem 4.1, we obtain a fibration $f : X \to \overline{Y}$ such that M_X descends to an ample bundle $M_{\overline{Y}}$, by Lemma 3.15. Since $M_{X\backslash D}$ is pulled back from Y and is strictly nef on Y it follows that $\overline{Y}\backslash f(D) \cong Y$ and hence that $\overline{Y}$ is a compactification of Y. We prove now that $\overline{Y} \cong \operatorname{Proj} C_Y$.

Indeed, first note that every element of $H^0_{\mathrm{mg}}(Y, M_Y^n)$ pulls back to an element of $H^0(X, M_X^n)$ and thus descends to an element of $H^0(\overline{Y}, M_{\overline{Y}}^n)$. Conversely, any element of $H^0(\overline{Y}, M_{\overline{Y}}^n)$ is a section of moderate growth, and thus belongs to C_Y. Hence, we see that $C_Y = \bigoplus_{n\geq 0} H^0(\overline{Y}, M_{\overline{Y}}^n)$, from which the claim follows, and (1) is proved. Part (2) is clear since a local generator of $M_{\overline{Y}}^n$ pulls back to a generator of M_X^n. As above, for (3) it suffices to argue that the subsheaf of moderate growth sections of $j_*(M_Y^n)$ is a line bundle provided local sections as in (2) exist, but this follows from the normality of $\overline{Y}$ since any moderate growth function on Y (locally on $\overline{Y}$) extends to $\overline{Y}$. $\square$

5.4. **Proof of Theorem 5.2 (1).** By the above reduction, we may assume Γ is neat. Let $\nu : Z \to Y$ be the normalization of Y. By applying Theorem 5.3 (using Lemma 2.19 and Theorem 2.22), we have a Baily–Borel compactification Z^{BB} satisfying the requirements of Theorem 5.2. Let $R_0 \coloneqq Z(\mathbb{C})\times_{Y(\mathbb{C})} Z(\mathbb{C}) \subset Z(\mathbb{C})\times Z(\mathbb{C})$ be the equivalence relation defining the map to $Y(\mathbb{C})$, let $\overline{R_0} \subset Z^{\mathrm{BB}}(\mathbb{C}) \times Z^{\mathrm{BB}}(\mathbb{C})$ be its closure, and $R = (\overline{R_0})^e \subset Z^{\mathrm{BB}}(\mathbb{C}) \times Z^{\mathrm{BB}}(\mathbb{C})$ the equivalence relation it generates. Let (X, D) be a proper strictly log smooth algebraic space (X, D) with a proper birational morphism $X\backslash D \to Y$, which necessarily factors through Z.

Definition 5.7. For a point $x \in X$, we refer to $H(x)$ as the triple of rational mixed Hodge structures and morphisms of rational mixed Hodge structures

$$\left({}_{\mathrm{orig}}V^{\mathrm{gr}}(x),\ V^{\mathrm{min},\vee}(x),\ \iota_x : \mathrm{gr}^W V^{\mathrm{min},\vee}(x) \hookrightarrow \bigotimes_p \bigwedge^{\mathrm{rk}\, F^p{}_{\mathrm{orig}}V} {}_{\mathrm{orig}}V^{\mathrm{gr}}(x)^\vee\right)$$

with the obvious notion of isomorphism $H(x) \to H(y)$, namely, isomorphisms ${}_{\mathrm{orig}}V^{\mathrm{gr}}(x) \to {}_{\mathrm{orig}}V^{\mathrm{gr}}(y)$ and $V^{\mathrm{min},\vee}(x) \to V^{\mathrm{min},\vee}(y)$ for which the induced maps commute with ι_x and ι_y.

The fiber $L(x)$ of the Griffiths bundle at x is realized as the Hodge line in $V^{\mathrm{min}}(x)$, and by Claim 2.24 and Lemma 2.25 any automorphism of $H(x)$ induces a torsion automorphism of $L(x)$, whose order is bounded by $\mathrm{rk}\,{}_{\mathrm{orig}}V$.

Corollary 5.8. *There exists a positive integer N such that for any $x, x' \in X$, there is at most one isomorphism $L^N(x) \to L^N(x')$ induced by an isomorphism $H(x) \to H(x')$.*

We obtain a natural equivalence relation R_H on X with $x \sim_H y$ if $H(x) \cong H(y)$. As in §3.3, we have $R_{\mathrm{curve}} \subset R_H \subset R_{\mathrm{tr}}$, and so R_H descends to $Z^{\mathrm{BB}}(\mathbb{C}) = X(\mathbb{C})/R_{\mathrm{curve}}$. We abusively use the same notation R_H for the equivalence relation on $Z^{\mathrm{BB}}(\mathbb{C})$.

Lemma 5.9. *The equivalence relation R is algebraic, finite, and $R \subset R_H$. In particular, R_H descends through the quotient $q : |Z^{\mathrm{BB}}| \to \overline{Y} \coloneqq Z^{\mathrm{BB}}(\mathbb{C})/R$.*

Proof. Let $R_X \subset X(\mathbb{C}) \times X(\mathbb{C})$ be the closure of the pullback of R_0; note that it surjects onto $\overline{R_0}$. For any $(x, y) \in R_X$, there is a curve $C \subset R_X \cap (X \backslash D) \times (X \backslash D)$ whose closure contains (x, y). The pullback of the variation to C under the two resulting maps $C \rightrightarrows X \backslash D$ are equal, as therefore are the limit mixed Hodge structures, so $H(x) \cong H(y)$. Thus, $R_{\text{curve}} \subset (R_X \cup R_{\text{curve}})^e \subset R_H \subset R_{\text{tr}}$, so by Lemma 3.4(4) $(R_X \cup R_{\text{curve}})^e$ is algebraic. Since $Z^{\text{BB}}(\mathbb{C}) = X(\mathbb{C})/R_{\text{curve}}$, the image of $(R_X \cup R_{\text{curve}})^e$ in $Z(\mathbb{C}) \times Z(\mathbb{C})$ is R, so it is algebraic as well, finite by Lemma 3.4(2), and contained in R_H (on Z^{BB}). $\square$

Observe that, given any definable disk[9] $\Delta^* \to \overline{Y}$ lifting to a definable analytic map $\Delta^* \to Z^{\text{BB}}$, and any two such lifts $f, f' : \Delta^* \to Z^{\text{BB}}$, any choice of isomorphism $H(f(t)) \cong H(f'(t))$ at a very general point $t \in \Delta^*$ extends to an isomorphism of the natural variation in the very general H data on Δ^*, up to shrinking Δ, which then gives an isomorphism $H(f(t)) \to H(f'(t))$ at every point $t \in \Delta$ via the limit mixed Hodge structure. Moreover, the induced isomorphism $L^N(f(t)) \to L^N(f'(t))$ is continuous for $t \in \Delta$.

Lemma 5.10. *There is a proper strictly log smooth algebraic space (X, D) and a proper birational morphism $X \backslash D \to Y$ such that:*

1. *(X, D) satisfies (B1), (B2), (B3)$_R$. In particular,*
   - (a) *The Hodge strata X_S of X are saturated with respect to $X \to Z^{\text{BB}}$ (resp. $|X| \to \overline{Y}$) and descend to strata Z^{BB}_S (resp. $\overline{Y}_S$) of Z^{BB} (resp. $\overline{Y}$).*
   - (b) *The R-strata X_T of X are saturated with respect to $X \to Z^{\text{BB}}$ (resp. $|X| \to \overline{Y}$) and descend to strata Z^{BB}_T (resp. $\overline{Y}_T$) of Z^{BB} (resp. $\overline{Y}$).*
2. *Each Hodge stratum Z^{BB}_S is smooth and each R-stratum Z^{BB}_T is a disjoint union of Hodge strata Z^{BB}_S.*
3. *For each Hodge stratum Z^{BB}_S, the π_1-definable analytic morphism*
$$\rho_{Z^{\text{BB}},S} : \widetilde{Z^{\text{BB}}_S}^{V^{\text{tr}}_{S,\mathbb{Q}}} \to \mathbb{P}(V^{\text{tr}}_{Z^{\text{BB}},S,\mathbb{C},z(S)})^{\text{an}}$$
obtained from projecting the Hodge bundle is unramified.

Proof. We essentially redo the last part of the proof of Lemma 3.7. Namely, we construct such a stratification by descending induction on the dimension of Z^{BB}_Σ where the lemma is false, the base case being trivial. Thus, assume the condition holds for any S with $\dim Z^{\text{BB}}_S > k$, and consider a stratum with $\dim Z^{\text{BB}}_S = k$. Let $W \subset Z^{\text{BB}}_S$ be the locus where either Z^{BB}_S isn't smooth or the period map isn't unramified. Note that $\dim W < k$ and consider the pullback Z to X of the R-saturation $R(\overline{W})$ of $\overline{W}$. We now pass to a log resolution of Z, and modify further as necessary for (X', D') satisfies (B3)$_R$. As in the proof of Lemma 3.7, only strata with period image of dimension strictly smaller than k are produced. On the other hand, the new stratum $Z^{\text{BB}}_S \backslash R(W)$ now satisfies the conditions. Continuing in this way, we are done by induction. $\square$

From the proof of Theorem 4.4, the variations of Hodge structures $V^{\min}_T$ and V^{tr}_T on an R-stratum X_T of X descend to $V^{\min}_{Z^{\text{BB}},T}$ and $V^{\text{tr}}_{Z^{\text{BB}},T}$ on the corresponding R-stratum Z^{BB}_T. We may take DR-neighborhoods $Z_T \subset \mathfrak{T}_{Z^{\text{BB}}}(T) \subset Z^{\text{BB}}$ and $\overline{Y}_T \subset \mathfrak{T}_{\overline{Y}}(T) \subset \overline{Y}$ of the R-strata in both Z^{BB} and $\overline{Y}$ such that each $\mathfrak{T}_{Z^{\text{BB}}}(T)$ (resp. $\mathfrak{T}_{\overline{Y}}(T)$) only intersects strata limiting to Z^{BB}_T (resp. $\overline{Y}_T$) and such that each $\mathfrak{T}_{Z^{\text{BB}}}(T)$ maps to $\mathfrak{T}_{\overline{Y}}(T)$.

Corollary 5.11. *The conclusions of Lemma 5.10 hold with respect to $E := \operatorname{Sym}^N V$ in place of V. Moreover, we have the following:*

1. *The Hodge bundle of E descends to $\overline{Y}$ as a continuous line bundle $L^{(N)}_{\overline{Y}}$.*

[9] It would suffice to use algebraic curves.

(2) $E^{\min}_{Z^{\mathrm{BB}},T}$ descends to the R-stratum $\overline{Y}_T$ as a rational local system whose fibers are continuously endowed with Hodge structures, $E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T}$ descends as a subobject in this category, and the Hodge line in both is identified with $L^{(N)}_{\overline{Y}}$. We call the resulting objects $E^{\min}_{\overline{Y},T}$ and $E^{\mathrm{tr}}_{\overline{Y},T}$.

(3) By projecting the Hodge bundle (of E) we have π_1-definable analytic

$$\tau(T) : \widetilde{\mathfrak{T}_{Z^{\mathrm{BB}}}(T)}^{E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T,\mathbb{Q}}} \to \mathbb{P}(E^{\min}_{\overline{Y},T,\mathbb{C},z(T)})^{\mathrm{an}}$$

which is unramified in restriction to $\widetilde{Z^{\mathrm{BB}}_T}^{E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T,\mathbb{Q}}}$ and pointwise factors through $\widetilde{\mathfrak{T}_{\overline{Y}}(T)}^{E^{\mathrm{tr}}_{\overline{Y},T,\mathbb{Q}}}$ (on the preimage of $\widetilde{\mathfrak{T}_{\overline{Y}}(T)}^{E^{\mathrm{tr}}_{\overline{Y},T,\mathbb{Q}}}$).

Proof. By Corollary 5.8, Lemma 5.9, and the observation before Lemma 5.10, R gives a continuous descent datum on the Hodge bundle of E (which is the Nth power of the Hodge bundle of V), which proves (1).

By Lemma 5.9 it follows that we obtain at every point of $\overline{Y}$ a well-defined isomorphism class of Hodge structures which are the descent of $E^{\min}_{Z^{\mathrm{BB}},T}$ and $E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T}$. By Corollary 5.8 and Lemma 2.10 there is a canonical descent datum which is pointwise induced by an isomorphism of H-data, and by the observation before Lemma 5.10 it is continuous, so (2) follows.

Part (3) is immediate from part (2) and part (3) of Lemma 5.10. □

Next, we show the objects $E^{\min,\vee}_{\overline{Y},T}$ are compatible between strata as rational local systems whose fibers are continuously endowed with Hodge structures. In the following, we abusively denote the pullback of R to X by the same letter. For any Hodge strata S_1, S_2 of X, the descent data for the $E^{\min,\vee}_{S_i}$ and their quotients $E^{\mathrm{tr},\vee}_{S_i}$ naturally gives via Lemma 2.7 an isomorphism of local systems $R_{S_1,S_2} : p_1^*E^{\min,\vee}(S_1)_{\mathbb{Q}} \to p_2^*E^{\min,\vee}(S_2)_{\mathbb{Q}}$ on $R \cap \mathfrak{T}(S_1) \times \mathfrak{T}(S_2)$ which is compatible with the corresponding morphism on the quotients to $p_i^*E^{\mathrm{tr},\vee}(S_i)_{\mathbb{Q}}$. Recall by Lemma 2.12 that if X_S specializes to $X_{S'}$, then $E^{\min,\vee}(S')_{\mathbb{Q}}$ is naturally a sub-local system of $E^{\min,\vee}(S)_{\mathbb{Q}}$ on the intersection $\mathfrak{T}_{Z^{\mathrm{BB}}}(S) \cap \mathfrak{T}_{Z^{\mathrm{BB}}}(S')$.

Lemma 5.12. *Let X_{S_1} (resp. X_{S_2}) be a Hodge stratum specializing to $X_{S_1'}$ (resp. $X_{S_2'}$). On $\big(\mathfrak{T}(S_1) \cap \mathfrak{T}(S_1')\big) \times \big(\mathfrak{T}(S_2) \cap \mathfrak{T}(S_2')\big)$ we have $R_{S_1,S_2} |_{p_1^*E^{\min,\vee}(S_1')_{\mathbb{Q}}} = R_{S_1',S_2'}$.*

Proof. Since these are maps of local systems, it is enough to check the statement at a single point in each connected component. Hence, let $\Delta^* \to \big(X_{S_1} \cap \mathfrak{T}(S_1')\big) \times \big(X_{S_2} \cap \mathfrak{T}(S_2')\big)$ be a definable analytic disk whose image is Zariski dense in $X_{S_1} \times X_{S_2}$, and which extends to a map $\Delta \to R$ with the origin landing in $X_{S_1'} \times X_{S_2'}$. By the observation before Lemma 5.10, we obtain an isomorphism in the resulting two variations of the very general H-data over Δ^* up to shrinking Δ, and therefore pointwise of the H-data at every point. These isomorphisms induce R_{S_1,S_2} on $E^{\min,\vee}_{S_1}$ over Δ^* and $R_{S_1',S_2'}$ on $E^{\min,\vee}_{S_1'}$ at $0 \in \Delta$ by Corollary 5.8 and Lemma 2.10. By Lemma 2.12 the claim follows. □

It follows that if $\overline{Y}_T$ specializes to $\overline{Y}_{T'}$, the local system $E^{\min,\vee}_{\overline{Y}}(T')_{\mathbb{Q}}$ is naturally a sub-local system of $E^{\min,\vee}_{\overline{Y}}(T)_{\mathbb{Q}}$ on $\mathfrak{T}_{\overline{Y}}(T) \cap \mathfrak{T}_{\overline{Y}}(T')$, and that the restriction of the quotient $E^{\min,\vee}_{\overline{Y}}(T)_{\mathbb{Q}} \to E^{\mathrm{tr},\vee}_{\overline{Y}}(T)_{\mathbb{Q}}$ factors through the quotient $E^{\min,\vee}_{\overline{Y}}(T')_{\mathbb{Q}} \to E^{\mathrm{tr},\vee}_{\overline{Y}}(T')_{\mathbb{Q}}$. Dually, $E^{\min}_{\overline{Y}}(T')_{\mathbb{Q}}$ is naturally a quotient of $E^{\min}_{\overline{Y}}(T)_{\mathbb{Q}}$ and the quotient map takes $E^{\mathrm{tr}}_{\overline{Y}}(T)_{\mathbb{Q}}$ to $E^{\mathrm{tr}}_{\overline{Y}}(T')_{\mathbb{Q}}$, again on $\mathfrak{T}_{\overline{Y}}(T) \cap \mathfrak{T}_{\overline{Y}}(T')$.

Finally, we give an algebraic structure to $\overline{Y}$. We shall follow Theorem 4.4, and so we build our algebraic structure one R-stratum at a time, inductively. We therefore let $U \subset \overline{Y}$ be an open union of R-strata, $\overline{Y}_T \subset U$ an R-stratum which is closed in U, and we inductively suppose that $U' := U \backslash \overline{Y}_T$ has been given an algebraic structure together with an algebraic map $Z^{\mathrm{BB}}_{U'} := q^{-1}(U') \to U'$ which are compatible with the definable topological space structures. We further suppose:

(i) The line bundle $\mathcal{O}_{Z^{\mathrm{BB}}}(N)$ of Z^{BB} restricted to $Z^{\mathrm{BB}}_{U'}$ descends[10] to an ample line bundle A' on U'.
(ii) For each R-stratum $\overline{Y}_{T'} \subset U'$, the morphism obtained by projecting the Hodge bundle

$$\tau(T') : \widetilde{\mathfrak{T}_{Z^{\mathrm{BB}}}(T')}^{E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T',\mathbb{Q}}} \to \mathbb{P}(E^{\min}_{Z^{\mathrm{BB}},T',\mathbb{C},z(T')})^{\mathrm{an}}$$

factors through $\widetilde{\mathfrak{T}_{\overline{Y}}(T')}^{E^{\mathrm{tr}}_{\overline{Y},T',\mathbb{Q}}}$.

The base case $U = Y$ is trivial given the above setup.

On the one hand, by Theorem 2.5 as in Lemma 3.15, we may pick a finite-dimensional homogeneous subspace of B_Y yielding a linear system of sections of a power of A' which extend to Z^{BB} and which embed U' in $\mathbb{P}^N$. On the other hand, by Corollary 5.11 and Lemma 5.12 we have π_1-definable analytic morphisms

$$\tau(T) : \widetilde{\mathfrak{T}_{Z^{\mathrm{BB}}}(T)}^{E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T,\mathbb{Q}}} \to \mathbb{P}(E^{\min}_{\overline{Y},T,\mathbb{C},z(T)})^{\mathrm{an}}$$

which is pointwise compatible with $\tau(T')$ for each stratum $Z^{\mathrm{BB}}_{T'} \subset Z^{\mathrm{BB}}_{U'}$ by the paragraph right after Lemma 5.12. Combining the resulting linear system for an appropriate power of $\mathcal{O}_{Z^{\mathrm{BB}}}(N)|_{\mathfrak{T}_{Z^{\mathrm{BB}}}(T)}$ with the previous one, we obtain a definable analytic morphism

$$\pi(T) : \widetilde{\mathfrak{T}_{Z^{\mathrm{BB}}}(T)}^{E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T,\mathbb{Q}}} \to (\mathbb{P}^{N_T})^{\mathrm{an}}$$

which factors through a local embedding of $\widetilde{U'} \cap \widetilde{\mathfrak{T}^*_{Z^{\mathrm{BB}}}(T)}^{E^{\mathrm{tr}}_{Z^{\mathrm{BB}},T,\mathbb{Q}}}$, and whose restriction to $\widetilde{Z^{\mathrm{BB}}_T}$ is both unramified and factors through $\widetilde{\overline{Y}_T}$ on $\widetilde{Z^{\mathrm{BB}}_T}$. Thus, it is everywhere locally injective and factors through $\widetilde{\mathfrak{T}_{\overline{Y}}(T)}$.

Now, observe that in the analytic (resp. definable analytic) category we have:

- Any morphism $f : X \to Y$ with discrete fibers factors as $X \to Z \to Y$ where $X \to Z$ is finite and $Z \to Y$ is an open embedding, up to replacing X with a cover. In the definable analytic category, this follows from [BBT24, Lemma 2.8].
- For a locally injective morphism $f : X \to Y$, a factorization $|X| \to \mathfrak{Z} \to |Y|$ with $|X| \to \mathfrak{Z}$ finite and surjective on the level of topological spaces (resp. definable topological spaces) can be uniquely lifted to a factorization $X \to Z \to Y$ for which $Z \to Y$ is unramified. Indeed, using the previous bullet point, and the fact that any cover of X can be refined by a cover consisting of the connected components of the pullback of a cover from $\mathfrak{Z}$ (which is [BBT23a, Proposition 2.4] in the definable analytic category), we may assume (after passing to a cover of $\mathfrak{Z}$) that on every connected component of X, $X \to Y$ is a homeomorphism followed by a locally closed embedding, and that the image is identified with $\mathfrak{Z}$. The sheaf of functions on $\mathfrak{Z}$ is then that of the image, using [BBT23a, Proposition 2.52][11] in the definable analytic category.

Applying the second bullet point above to $\pi(T)$ as in Claim 4.5, it follows that there is a definable analytic space structure on $\mathfrak{T}_{\overline{Y}}(T)$ and a morphism of definable analytic spaces $\mathfrak{T}_{Z^{\mathrm{BB}}}(T) \to \mathfrak{T}_{\overline{Y}}(T)$ whose underlying map on definable topological spaces is the quotient map, and therefore there is a definable analytic space structure on U and a morphism of definable analytic spaces $(Z^{\mathrm{BB}}_U)^{\mathrm{def}} \to U$ whose underlying map is the quotient map and which is compatible with $(Z^{\mathrm{BB}}_{U'})^{\mathrm{def}} \to U'^{\mathrm{def}}$. By the definable image theorem (Theorem 2.2), the definable analytic space structure on U is (uniquely) algebraizable, as is the morphism $f_U : Z^{\mathrm{BB}}_U \to U$. By construction, (ii) is satisfied. Also by construction, $\mathcal{O}_{Z^{\mathrm{BB}}_U}(N)$ descends to a definable analytic line bundle on U which is naturally contained in $(f_{U*}\mathcal{O}_{Z^{\mathrm{BB}}_U}(N))^{\mathrm{def}}$, hence algebraic by definable GAGA (Theorem 2.1),

[10] Note that we already established this descent as a continuous line bundle, but we want it as an algebraic line bundle.

[11] As we are only concerned with reduced spaces, [BBT23a, Proposition 2.45] would suffice.

and ample by Theorem 2.5. Thus, by induction there is an algebraic space structure on $\overline{Y}$ and a morphism $Z^{\mathrm{BB}} \to \overline{Y}$ (whose underlying map is the quotient map) such that $\mathcal{O}_{Z^{\mathrm{BB}}}(N)$ descends to an ample bundle $L_{\overline{Y}}^{(N)}$.

To conclude, it follows that $B_{\overline{Y}} := \bigoplus_{k\geq 0} H^0(\overline{Y}, L_{\overline{Y}}^{(kN)}) \subset B_Y \subset B_{Z^{\mathrm{BB}}}$. Since B_Y is a submodule of the finitely generated $B_{\overline{Y}}$-module $B_{Z^{\mathrm{BB}}}$ it follows that B_Y is finitely generated, so we may define $Y^{\mathrm{BB}} := \operatorname{Proj} B_Y$, and it follows that $Y \hookrightarrow Y^{\mathrm{BB}}$ since $B_{\overline{Y}}$ and hence B_Y induces an embedding of Y. □

5.5. **Proof of Theorem 5.2 (2) and (3).** Again by the above reduction, we may assume Γ is neat. We first prove part (2). Let (X, D) be a log smooth algebraic space with a proper birational morphism $X\backslash D \to Y$. By construction, some power $L_X^{(n)}$ descends to $L_{Y^{\mathrm{BB}}}^{(n)}$ as a line bundle. It follows that any locally generating section s on Y^{BB} pulls back to a generating section on X and thus has Hodge norm and inverse Hodge norms of moderate growth. Thus, the set of k_Y in question is nonempty. Since it clearly forms a group, there is a minimal one and (2) follows.

We now prove (3):

Lemma 5.13. *Consider the maps $\overline{\nu} : Z^{\mathrm{BB}} \to Y^{\mathrm{BB}}$, $j : Y \to Y^{\mathrm{BB}}$. Then we have the equality $\mathcal{O}_{Y^{\mathrm{BB}}} = j_*\mathcal{O}_Y \cap \overline{\nu}_*\mathcal{O}_{Z^{\mathrm{BB}}}$, with the intersection taking place in $j_*\nu_*\mathcal{O}_Z$. Moreover, $\mathcal{O}_{Y^{\mathrm{BB}}}$ analytifies to $j_*^{\mathrm{an}}\mathcal{O}_{Y^{\mathrm{an}}} \cap \overline{\nu}_*^{\mathrm{an}}\mathcal{O}_{Z^{\mathrm{BB,an}}}$.*

Proof. Let $\mathcal{R} = j_*\mathcal{O}_Y \cap \overline{\nu}_*\mathcal{O}_{Z^{\mathrm{BB}}}$. It is clear that $\mathcal{R}$ is quasicoherent, and since it injects into $\overline{\nu}_*\mathcal{O}_{Z^{\mathrm{BB}}}$ it must be coherent.

Now consider $W = \operatorname{Spec} \mathcal{R}$. By construction W fits into a map $Z^{\mathrm{BB}} \to W \to Y^{\mathrm{BB}}$. Hence, some power the Griffiths bundle $L^{(n)}$ descends to W as $L_W^{(n)}$, and therefore is ample there. Thus $W = \operatorname{Proj} B_W$. However, clearly $B_W \subset B_Y$, and thus we must have equality. It follows that $W = Y^{\mathrm{BB}}$ which completes the proof.

Finally, the analytification statement would follow directly from (ordinary) GAGA if it weren't for the fact that $j_*\mathcal{O}_Y$ is quasicoherent as opposed to coherent. To address that, we work locally and let h be a regular function on Y^{BB} vanishing on the boundary. For $m \geq 1$ let $\mathcal{R}_m := h^{-m}\mathcal{O}_Y \cap \overline{\nu}_*\mathcal{O}_{Z^{\mathrm{BB}}}$. It is clear that $\mathcal{R}_m$ analytifies to $(\mathcal{R}^{\mathrm{an}})_m := h^{-m}\mathcal{O}_{Y^{\mathrm{an}}} \cap \overline{\nu}_*^{\mathrm{an}}\mathcal{O}_{Z^{\mathrm{BB,an}}}$, and so the claim follows as $j_*^{\mathrm{an}}\mathcal{O}_{Y^{\mathrm{an}}} \cap \overline{\nu}_*^{\mathrm{an}}\mathcal{O}_{Z^{\mathrm{BB,an}}} = \cup_m(\mathcal{R}^{\mathrm{an}})_m$. □

We may define a coherent sheaf $L_{Y^{\mathrm{BB}}}^{(k_Y)} \subset j_*L_Y^{(k_Y)}$ by considering all local sections whose Hodge norms have moderate growth.

Corollary 5.14. *$L_{Y^{\mathrm{BB}}}^{(k_Y)}$ is a line bundle on Y^{BB}.*

Proof. We work locally around a point $y \in Y^{\mathrm{BB}}$. By part (2), there is an affine neighborhood $y \in U$ and a local section $s \in H^0(U, L_{Y^{\mathrm{BB}}}^{(k_Y)})$ whose Hodge norm and its inverse have moderate growth around every point in $U\backslash Y$. We claim that s is a local generator around y.

Suppose that $s' \in H^0(U, L_Y^{(k_Y)})$ is some other moderate growth section. Then $t = s'/s \in H^0(U \cap Y, \mathcal{O}_{Y^{\mathrm{BB}}})$ has moderate growth, hence is bounded locally on $U\backslash Y$. Since Z^{BB} is normal, it follows that ν^*t extends to a function on $\operatorname{int}\left(\overline{\overline{\nu}^{-1}(U)}\right) = \overline{\nu}^{-1}(\operatorname{int}(\overline{U})))$ since $\overline{\nu} : Z^{\mathrm{BB}} \to Y^{\mathrm{BB}}$ is open. Hence t extends to an element of $H^0(U, \mathcal{O}_{Y^{\mathrm{BB}}})$ by Lemma 5.13 as desired. □

Finally, we complete the proof of (3). It is clear that $L_Y^{(k_Y)}$ is ample. Now it follows from Corollary 5.14 that we have $H^0(Y^{\mathrm{BB}}, L_{Y^{\mathrm{BB}}}^{(k_Y)}) = H^0_{\mathrm{mg}}(Y, L_Y^{(k_Y)})$, and thus $L_{Y^{\mathrm{BB}}}^{(k_Y)}$ is naturally identified with $\mathcal{O}_{Y^{\mathrm{BB}}}(k_Y)$ as desired. □

5.6. **Proof of Theorem 5.5.** The compatibility with the Schmid extensions is immediate from parts (2) and (3) of Theorem 5.2 (resp. Theorem 5.3), so we focus on the existence on the extension of the morphism. The proof for Y^{BBH} is the same so we focus on the Y^{BB} statement. Let $f : Y' \to Y$ be a finite étale cover of Y with level structure so that the monodromy group is neat. There is then a finite map $\pi : \Delta^k \to \Delta^k : (z_1, \dots, z_k) \mapsto (z_1^N, \dots, z_k^N)$ and a commutative diagram

$$\begin{tikzcd} (\Delta^*)^k \arrow[r] \arrow[d, "\pi|_{(\Delta^*)^k}"'] & Y' \arrow[d] \\ (\Delta^*)^k \arrow[r] & Y \end{tikzcd}$$

and it is sufficient to show the top map extends. Thus, we may assume the variation has neat monodromy.

The extension of ϕ is unique if it exists, so the claim is local on $(\Delta^*)^k$, and we may freely shrink Δ^k. Thus, we may assume $\phi : (\Delta^*)^k \to Y$ is definable (by [BKT20, Theorem 4.1][12]) and therefore that it extends meromorphically, as in [BBT23b, Lemma 2.2]. By Hironaka's embedded resolution theorem, we may construct a tower of blowups along smooth centers

$$X_r \to X_{r-1} \cdots \to X_0 = \Delta^k$$

such that ϕ extends to a morphism $\phi_r : X_r \to Y^{\mathrm{BB,an}}$ and the pair (X_r, D_r) is log smooth, where D_r is the union of exceptional divisors and the strict transform of the coordinate hyperplanes of Δ^k. Locally on Y^{BB}, the Griffiths bundle has a generating section with moderate growth and whose inverse has moderate growth, and it follows that the pullback $\phi_r^* L_{Y^{\mathrm{BB}}}$ agrees with the Schmid extension of the Griffiths bundle of the variation on $X_r \backslash D_r$. But then we also have $\phi_r^* L_{Y^{\mathrm{BB}}} \cong f^* L_{\Delta^k}$ where $f : X_r \to \Delta^k$ is the blow-down and L_{Δ^k} is the Schmid extension of the variation on $(\Delta^*)^k$. Thus, $\phi_r^* L_{Y^{\mathrm{BB}}}$ is trivial on every fiber of f, hence ϕ_r factors through f, as desired. $\square$

5.7. **Proof of Theorem 5.2 (4) and Theorem 5.3 (4).** The existence of the extension of the morphism and the compatibility with the Schmid extension is immediate from Theorem 5.5. The uniqueness statement in Theorem 5.3(4) is standard: if Z were another such compactification of Y, then $X \backslash D \to Y$ also extends to $X \to Z$, but since $\mathcal{O}_Z(n)$ and $\mathcal{O}_{Y^{\mathrm{BBH}}}(n)$ are both ample and pullback to the same line bundle on X, $X \to Y^{\mathrm{BBH}}$ factors through Z and $X \to Z$ factors through Y^{BBH} by normality, hence $Z \cong Y^{\mathrm{BBH}}$. $\square$

6. Birational geometry and Hodge theory of lc-trivial fibrations

The moduli part of an Ambro model is the Hodge bundle of a variation of Hodge structure. We discuss in detail the variation arising (Construction-Definition 6.14), and provide a geometric characterization of its restriction in codimension one in terms of sources of slc pairs (cf. Theorem 6.31). To this end, we first recall the notions of b-divisor, pairs, canonical bundle formula, and locally stable families. We refer to [KM98] and [Kol13] for the standard terminology in birational geometry.

6.1. **B-divisors.** Let $\mathbb{K}$ denote $\mathbb{Z}$ or $\mathbb{Q}$. Given a normal algebraic space X, a $\mathbb{K}$-*b-divisor* $\mathbf{D}$ is a (possibly infinite) sum of geometric valuations ν_i of $k(X)$ with coefficients in $\mathbb{K}$,

$$\mathbf{D} = \sum_{i \in I} b_i \nu_i, \ b_i \in \mathbb{K},$$

[12] The statement therein should read that the local period map is definable *up to shrinking* Δ^k.

such that, given any normal variety X' birational to X, only finitely many valuations ν_i have a center of codimension 1 on X'. The *trace* $\mathbf{D}_{X'}$ of $\mathbf{D}$ on Y' is the $\mathbb{K}$-Weil divisor

$$\mathbf{D}_{Y'} \coloneqq \sum b_i D_i$$

where the sum is indexed over valuations ν_i that have divisorial center $D_i \subset X'$.

Given a $\mathbb{K}$-b-divisor $\mathbf{D}$ over X, we say that $\mathbf{D}$ is a $\mathbb{K}$-*b-Cartier* if there exists a birational model X' of X such that $\mathbf{D}_{X'}$ is $\mathbb{K}$-Cartier on X' and for any model $\pi\colon X'' \to X'$, we have $\mathbf{D}_{X''} = \pi^*\mathbf{D}_{X'}$. When this is the case, we will say that $\mathbf{D}$ *descends* to X' and we shall write $\mathbf{D} = \overline{\mathbf{D}}_{X'}$ for the $\mathbb{K}$-b-divisor which $\mathbf{D}_{X'}$ determines. We say that $\mathbf{D}$ is *b-effective*, if $\mathbf{D}_{X'}$ is effective for any model X'. We say that $\mathbf{D}$ is *b-nef* (resp. *b-semiample*), if it is $\mathbb{K}$-b-Cartier and, moreover, there exists a birational model X' of X such that $\mathbf{D} = \overline{\mathbf{D}}_{X'}$ and $\mathbf{D}_{X'}$ is nef (resp. semiample) on X'.

In all of the above, if $\mathbb{K} = \mathbb{Z}$, we will systematically drop it from the notation.

Example 6.1. Let (X, Δ) be a log sub-pair. The *discrepancy b-divisor* $\mathbf{A}(X, \Delta)$ is defined as follows: on a birational model $\pi : X' \to X$, its trace $\mathbf{A}(X, \Delta)_{X'}$ is given by the identity $\mathbf{A}(X, \Delta)_{X'} \coloneqq K_{X'} - \pi^*(K_X + \Delta)$. The b-divisor $\mathbf{A}^*(X, \Delta)$ is defined by taking its trace $\mathbf{A}^*(X, \Delta)_{X'}$ on X' to be

$$\mathbf{A}^*(X, \Delta)_{X'} \coloneqq \mathbf{A}(X, \Delta)_{X'} + \sum_{a(D_i;X,\Delta)=-1} D_i,$$

where $\mathbf{A}(X, \Delta)_{X'} = \sum_i a(D_i; X, \Delta) D_i$.

6.2. **Singularities of pairs.** The acronyms klt, dlt, lc, sdlt, and slc describe types of singularities that occur naturally in various constructions within birational geometry. For instance, the minimal (resp. canonical) model of an snc pair has dlt (resp. lc) singularities. The reduced part of the boundary of a dlt (resp. lc) pair is sdlt (resp. slc). The fibers of a semistable or locally stable morphism (e.g., the families of varieties parametrized by KSBA moduli spaces) have slc singularities. Finally, up to finite base change, any family of Calabi–Yau varieties over a punctured disk has a dlt log Calabi–Yau filling; see [Fuj11]. Here, we limit ourselves to recalling the relevant definitions and mentioning some properties of lc centers used in the following sections.

Definition 6.2 (Singularities of normal pairs)**.** Let (X, Δ) be a log sub-pair where X is a normal algebraic space.

- (X, Δ) is *Kawamata log terminal (klt)*, if $\lceil \mathbf{A}(X, \Delta) \rceil \geq 0$, i.e., $a(D; X, \Delta) > -1$ for every divisor D.
- (X, Δ) is *log canonical (lc)*, if $\lceil \mathbf{A}^*(X, \Delta) \rceil \geq 0$, i.e., $a(D; X, \Delta) \geq -1$ for every divisor D.
- (X, Δ) is *purely log terminal (plt)*, if $a(D; X, \Delta) > 0$ for every exceptional divisor D.
- An irreducible subvariety $Z \subset X$ of an lc sub-pair (X, Δ) is an *lc center* if there exist a birational morphism $\pi\colon X' \to X$ and a divisor $E \subset X$ with $a(E; X, \Delta) = -1$ whose image coincides with Z.
- (X, Δ) is *divisorial log terminal (dlt)* if (X, Δ) is lc and none of its lc centres lies in the complement of the largest open locus where the sub-pair is snc.

Definition 6.3 (Singularities of demi-normal pairs)**.** Let (X, Δ) be a log sub-pair where X is demi-normal, i.e., satisfies Serre's condition S_2 and it is nodal in codimension 1. Let $\nu\colon (\overline{X}, \overline{\Delta} + \overline{C}) \to (X, \Delta)$ be the normalization of (X, Δ) with conductor $\overline{C}$ and $\overline{\Delta} \coloneqq \nu^{-1}(\Delta)$.

- (X, Δ) is *semi-log canonical (slc)*, if $(\overline{X}, \overline{\Delta} + \overline{C})$ is lc.
- (X, Δ) is *semi-divisorial log terminal (sdlt)*, if (X, Δ) is slc and none of its lc centres lies in the complement of the largest open locus where the sub-pair is semi-snc.

Definition 6.4. A *log Calabi–Yau pair* (X, Δ) is a proper lc pair with $K_X + \Delta \sim_{\mathbb{Q}} 0$.

All minimal lc centers of a dlt log Calabi–Yau pair are $\mathbb{P}^1$-linked in the following sense.

Definition 6.5 (Standard $\mathbb{P}^1$-link)**.** A *standard* $\mathbb{P}^1$*-link* is a $\mathbb{Q}$-factorial pair $(X, D_1 + D_2 + \Delta)$ together with a proper morphism $\pi : X \to T$ such that:

(1) $K_X + D_1 + D_2 + \Delta \sim_{\mathbb{Q},\pi} 0$,
(2) $(X, D_1 + D_2 + \Delta)$ is plt (in particular, D_1 and D_2 are disjoint),
(3) the morphisms $\pi|_{D_1} : D_1 \to T$ and $\pi|_{D_2} : D_2 \to T$ are isomorphisms, and
(4) every reduced fiber X_t^{red} is isomorphic to $\mathbb{P}^1$.

Remark 6.6. Alternatively, the total space X of a standard $\mathbb{P}^1$-link is the projectivization of a split $\mathbb{Q}$-vector bundle of rank 2, whose two direct summands correspond to the sections D_1 and D_2; see [Mor24, Thm. 1.4].

Definition 6.7 ($\mathbb{P}^1$-linking)**.** Let $f\colon (Y, \Delta) \to X$ be a fibration such that (Y, Δ) is a dlt pair and $K_Y + \Delta \sim_{f,\mathbb{Q}} 0$, and let $Z_1, Z_2 \subset X$ be two lc centers.

- Z_1 and Z_2 are *directly* $\mathbb{P}^1$*-linked* if there exists an lc center $W \subset X$ containing both Z_i such that $f(W) = f(Z_1) = f(Z_2)$, and the pair $(W, \mathrm{Diff}^*_W(\Delta))$ (cf. [Kol13, §4.18]) is birational to a standard $\mathbb{P}^1$-link, with Z_i mapping to D_i. Observe that $W = X$ is allowed.
- Z_1 and Z_2 are $\mathbb{P}^1$*-linked* if there exists a sequence of lc centers $Z'_1, Z'_2, \ldots, Z'_m$ such that $Z'_1 = Z_1$, $Z'_m = Z_2$, and for each $i = 1, \ldots, m-1$, the centers Z'_i and Z'_{i+1} are directly $\mathbb{P}^1$-linked (or $Z_1 = Z_2$).

In particular, every $\mathbb{P}^1$-linking defines a crepant birational map between the pairs $(Z_1, \mathrm{Diff}^*_{Z_1}(\Delta))$ and $(Z_2, \mathrm{Diff}^*_{Z_2}(\Delta))$.

Proposition 6.8. [Kol13, Thm. 4.40] *Let $f\colon (Y, \Delta) \to X$ be a projective fibration such that (Y, Δ) is a dlt pair and $K_Y + \Delta \sim_{f,\mathbb{Q}} 0$. All minimal lc centers of (Y, Δ) among those that intersect a fixed fiber of f are $\mathbb{P}^1$-linked.*

Among the minimal lc centers in Proposition 6.8, those dominating X are of particular interest, and they are called sources of $f\colon (Y, \Delta) \to X$.

Definition 6.9 (Sources)**.** Let $f\colon (Y, \Delta) \to X$ be a fibration from an slc pair (Y, Δ) to an integral base X with $K_Y + \Delta \sim_{\mathbb{Q},f} 0$. A *source* of $f\colon (Y, \Delta) \to X$ is a generically klt pair obtained as an lc center, minimal among those dominating X, of a dlt modification $(Y^{\mathrm{dlt}}, \Delta^{\mathrm{dlt}})$ of the normalization $(\overline{Y}, \overline{\Delta} + \overline{C})$ of (Y, Δ):

$$(S, \Delta_S) \overset{\iota}{\hookrightarrow} (Y^{\mathrm{dlt}}, \Delta^{\mathrm{dlt}}) \xrightarrow{\pi} (\overline{Y}, \overline{\Delta} + \overline{C}) \xrightarrow{\nu} (Y, \Delta). \tag{6.1}$$

It is unique up to crepant birational equivalence; see [Kol13, §4.5].

6.3. **Canonical bundle formula.**

We recall the notion of lc-trivial fibration.

Definition 6.10. Let (Y, Δ) be a sub-pair with coefficients in $\mathbb{Q}$. A projective fibration $f\colon Y \to X$ is *lc-trivial* if

(i) (Y, Δ) is an lc sub-pair over the generic point of X;
(ii) $\mathrm{rank} f_*\mathcal{O}_Y(\lceil \mathbf{A}^*(Y, \Delta) \rceil) = 1$;
(iii) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on X such that $K_Y + \Delta \sim_{\mathbb{Q}} f^*L$.[13]

Remark 6.11. Note that property (ii) holds automatically if the general fiber $(Y_{\nu_X}, \Delta_{\nu_X})$ of f is a klt pair.

The canonical bundle formula is a broad term designating a formula for the $\mathbb{Q}$-divisor L in (iii), encoding the log canonical thresholds of the codimension one singularities of f (boundary part) and the variation of the general fiber (moduli part).

[13] Observe that lc-trivial stands for (relatively) trivial log canonical divisor, i.e., assumption (iii), and not to the type of the singularities in assumption (i).

Theorem 6.12 (Canonical bundle formula)**.** *Let $f\colon (Y,\Delta)\to X$ be an lc-trivial fibration. Then*

$$K_Y+\Delta\sim_{\mathbb{Q}} f^*(K_X+B_X+\mathbf{M}_X) \tag{6.2}$$

where B_X and $\mathbf{M}_X$ are the boundary *and* moduli *b-divisors. Furthermore, $(X,B_X,\mathbf{M})$ has the structure of a generalized sub-pair (cf. [BZ16], [FS23]). We say that $f\colon (Y,\Delta)\to X$ induces $(X,B_X,\mathbf{M})$.*

When (Y,Δ) is a klt (resp. lc) pair, then $(X,B_X,\mathbf{M})$ is a klt (resp. lc) generalized pair.

We recall the definition of B_X and $\mathbf{M}$. For further details, we refer to [Amb04, Amb05, FG14b, Kol07].

Definition 6.13 (Boundary divisor)**.** Let $f\colon (Y,\Delta)\to X$ be an lc-trivial fibration. The *boundary divisor* B_X is the $\mathbb{Q}$-divisor whose coefficient along the prime divisor D is given by

$$\operatorname{ord}_D(B_X)=\sup_E\left\{1-\frac{1+a(E;Y,\Delta)}{\operatorname{mult}_E(\pi^*D)}\right\},$$

where the supremum is taken over all the divisors E over Y which dominate D, and $a(E;Y,\Delta)$ is the discrepancy of E with respect to (X,Δ).

Construction-Definition 6.14 (Moduli part)**.** Let $f\colon (Y,\Delta)\to X$ be a lc-trivial fibration of relative dimension n. Write Δ as difference of effective divisors without common components, namely

$$\Delta\coloneqq E+F-G\quad\text{with}\quad E\coloneqq\Delta^{=1}, G\coloneqq\lceil\Delta^{<0}\rceil$$

so that F is the fractional part of Δ satisfying $\lfloor F\rfloor=0$.

(†) Suppose that there exists an snc pair (X,D) such that (Y°,Δ°) is a locally trivial snc pair over $X^\circ\coloneqq X\setminus D$ with $K_{Y^\circ}+\Delta^\circ\sim_{\mathbb{Q},f}0$, where the superscript $^\circ$ refers to the restriction of an object of interest over X°.

Let d be the minimal positive integer such that dF° is an integral divisor and $d(K_{Y^\circ}+\Delta^\circ)\sim_f 0$. Consider $L\coloneqq\mathcal{O}_{Y^\circ}(G^\circ-K_{Y^\circ}-E^\circ)$. The isomorphism $L^d\simeq\mathcal{O}_{Y^\circ}(dF^\circ)$ determines a normalized cyclic cover $a\colon Y_2^\circ\to Y^\circ$ of degree d branched along F°, with quotient singularities; see [KM98, 2.49-53]. Choose a μ_d-equivariant resolution of singularities $h\colon Y_3^\circ\to Y_2^\circ$. Write $f_2\coloneqq f\circ a$ and $f_3\coloneqq f_2\circ h$, $E_2\coloneqq\operatorname{Supp}(a^{-1}E)$, $E_3^\circ\coloneqq\operatorname{Supp}(h^{-1}E_2^\circ)$.

$$\begin{tikzcd}
(Y^\circ,\Delta^\circ) \arrow[d, "f"'] & Y_2^\circ \arrow[l, "a"'] \arrow[dl, "f_2"] & Y_3^\circ \arrow[l, "h"'] \arrow[dll, "f_3"] \\
X^\circ. & &
\end{tikzcd}$$

The sheaf $(a\circ h)_*(\omega_{Y_3^\circ}(E_3^\circ))$ is μ_d-invariant, and admits a decomposition into μ_d-isotypic components

$$(a\circ h)_*(\omega_{Y_3^\circ}(E_3^\circ))\simeq a_*(\omega_{Y_2^\circ}(E_2^\circ))\simeq\bigoplus_{i=0}^{d-1}a_*(\omega_{Y_2^\circ}(E_2^\circ))_{\chi^i}\simeq\bigoplus_{i=0}^{d-1}\omega_{Y^\circ}\otimes L^i(-\lfloor i\Delta^\circ\rfloor),$$

where χ is a generator of the character group $\hat{\mu}_d$. In particular, we get the direct summand

$$(a\circ h)_*(\omega_{Y_3^\circ}(E_3^\circ))_\chi\simeq\omega_{Y^\circ}\otimes L\simeq\mathcal{O}_{Y^\circ}(G^\circ-E^\circ). \tag{6.3}$$

The χ-isotypic component of the restriction over X° of $R^n(f_3)_*\mathbb{C}_{Y_3^\circ\setminus E_3^\circ}$ determines a complex CY variation of mixed Hodge structures whose deepest nonzero piece of the Hodge filtration is the line bundle $f_*\mathcal{O}_{Y^\circ}(G^\circ-E^\circ)$; see (6.3) and Definition 6.10.(ii).

(††) Suppose that $R^n(f_3)_*\mathbb{C}_{Y_3^\circ\setminus E_3^\circ}$ has local unipotent monodromy.

Let V_Y be the Deligne/Schmid extension on X of the variation of mixed[14] Hodge structures $R^n(f_3)_*\mathbb{Q}_{Y_3^\circ \setminus E_3^\circ}$ defined on X°. Then the moduli part $\mathbf{M}_X$ is the Hodge bundle of the transcendental part of its χ-isotypic component

$$\mathbf{M}_X \coloneqq F^m(V_{Y,\mathcal{O}})^{\mathrm{tr}}_\chi.$$

Up to shifting the Hodge filtration of V_Y, the moduli part $\mathbf{M}_X$ is also the Hodge bundle of an admissible graded polarizable rational (not just complex!) CY variation V'_Y of mixed Hodge structures, whose underlying local system is the Deligne extension of $R^n(f_3)_*\mathbb{Q}_{Y_3^\circ \setminus E_3^\circ}$; see Remark 6.17.

If conditions (†) and (††) are not satisfied, then there exists a projective alteration $q\colon W \to X$ such that the pullback of the generic fiber of f along q extends to a fibration $f'\colon (Y', \Delta') \to W$ satisfying conditions (†) and (††) (cf. Proposition 6.27). Set

$$\mathbf{M}_X \coloneqq \frac{1}{\deg(q)} q_*(\mathbf{M}_W).$$

Example 6.15. If $f\colon Y \to X$ is a family of smooth CY varieties of dimension n, then $\mathbf{M}_X = f_*\omega_{Y/X}$ is simply the Hodge bundle of the variation of Hodge structures $R^n f_* \mathbb{Q}_Y$.

Remark 6.16. By the functoriality of Deligne/Schmid extension, $\mathbf{M}(f)$ pulls back to $\mathbf{M}(f')$ as a b-divisor, which descends on W because of the snc assumptions Construction-Definition 6.14; see [Kol07, §8.4.8].

Remark 6.17. $V_Y = (V_{Y,\mathbb{Q}}, W_\bullet V_{Y,\mathbb{Q}}, F^\bullet V_{Y,\mathcal{O}})$ is a graded polarizable rational mixed variation of Hodge structures of Hodge-level n but not CY in general, while the variation $(V_{Y,\mathbb{C}})_\chi$ is a complex CY variation, not rational if $d > 2$. Up to shifting the Hodge filtration of the μ_d-isotypic components, there exists a polarizable rational pure CY variation of Hodge-level $n+4$ whose deepest piece of the Hodge filtration is $\mathbf{M}_X$. Choose for instance

$$V'_Y = (V_Y)_\chi(2,-2) \oplus \bigoplus_{i \not\equiv 1,-1 (d)} (V_Y)_{\chi^i} \oplus (V_Y)_{\chi^{-1}}(-2,2).$$

In particular, note that the period map of V_Y is generically injective if and only if so is the period map of V'_Y.

Remark 6.18. Since Y_2° has quotient singularities (hence it is a rational homology manifold), the constant sheaf $\mathbb{C}_{Y_2^\circ}$ is a direct summand of $Rh_*\mathbb{C}_{Y_3^\circ}$. Its ($\mu_d$-equivariant) pushforward along a decomposes in isotypic components as follows

$$a_*\mathbb{C}_{Y_2^\circ} = \bigoplus_{i=0}^{d-1} \iota_* L_{\chi^i}, \tag{6.4}$$

where L_{χ^i} are suitable local systems on $\iota\colon Y^\circ \setminus F^\circ \hookrightarrow Y^\circ$, on which μ_d acts via the character χ^i. Then there exist isomorphisms of complex variations of Hodge structures

$$(R^n(f_3)_*\mathbb{C}_{Y_3^\circ \setminus E_3^\circ})^{\mathrm{tr}}_\chi \simeq (R^n(f_2)_*\mathbb{C}_{Y_2^\circ \setminus E_2^\circ})^{\mathrm{tr}}_\chi \simeq (R^n f_*(\iota_* L_\chi))^{\mathrm{tr}},$$

which yield equivalent alternative definitions of the moduli part. Compare the previous chain of isomorphisms with the different period maps considered in [Amb05]. This also fixes a minor inaccuracy in the definition of the variation of Hodge structures whose bottom piece extends to define the moduli part appearing in [Kol07, Def. (8.4.6)].

Remark 6.19. The $\mathbb{Q}$-linear equivalence in the canonical bundle formula (Theorem 6.12) can be upgraded to the linear equivalence of $\mathbb{Q}$-divisors $d(K_Y + \Delta) \sim d(f^*(K_X + B_X + \mathbf{M}_X))$, where d is the index appearing in Construction-Definition 6.14; see [PS09, §7.5].

[14]pure if $E^\circ = 0$, i.e., (Y, Δ) is klt over the generic point of X.

Remark 6.20. The canonical bundle formula (Theorem 6.12) continues to hold for projective fibrations $f\colon (Y,\Delta)\to X$ of complex analytic spaces satisfying (i), (ii), (iii) in Definition 6.10. Indeed, resolutions of singularities needed to achieve (†) and the cyclic covers in [KM98, 2.49-53] can be performed in the analytic category too. To achieve (††), we can take a suitable étale cover of $X\setminus D$ making the local monodromy of the relevant local system unipotent and then extend it to a finite cover (with quotient singularities) by the Grauert–Remmert Extension Theorem [GR03, Ch. XII, Thm. 5.4, p. 340]. Finally, the proof of [Kol07, Thm. 8.5.1] works through verbatim in the analytic context too.

Furthermore, if (Y,Δ) is klt over the generic point on X, the projectivity of f can be even replaced with the assumption that the morphism f is Kähler. In general, the projectivity assumption grants the polarizability of the rational variation of Hodge structures $R^n(f_3)_*\mathbb{Q}_{Y_3^\circ\setminus E_3^\circ}$, but in the klt case (i.e., $E_3^\circ=0$) such polarization is induced simply by the intersection form of the smooth proper fibers of f_3, regardless of their projectivity. It is unclear whether the projectivity of f is needed in the lc case. Note also that our proofs of Theorem 1.5 and Theorem 7.3 require the projectivity of the morphism.

6.4. Locally stable families.

Definition 6.21. Let X be a reduced scheme, $f\colon Y\to X$ a flat morphism of finite type and $f\colon (Y,\Delta)\to X$ a well-defined family of pairs (see [Kol23, Thm.-Def. 4.7]). Assume that (Y_x,Δ_x) is slc for every $x\in X$. Then $f\colon (Y,\Delta)\to X$ is *locally stable* if the following equivalent conditions hold:

(1) $K_{Y/X}+\Delta$ is $\mathbb{Q}$-Cartier;
(2) $f_T\colon (Y_T,\Delta_T)\to T$ is locally stable whenever T is the spectrum of a DVR and $q\colon T\to X$ is a morphism (see [Kol23, Thm.-Def. 2.3] for the notion of a locally stable family over a DVR).

We recall some properties of locally stable families.

Lemma 6.22 ([Kol23, Thm. 4.8])**.** *Let $f\colon (Y,\Delta)\to X$ be a locally stable morphism over a reduced base X, and $q\colon V\to X$ be a morphism of reduced schemes. Then the family over V obtained by fiber product is locally stable.*

Lemma 6.23 ([Kol23, Thm. 4.55])**.** *Let $f\colon (Y,\Delta)\to X$ be a morphism over a smooth scheme X with $\Delta\geq 0$. Then f is locally stable if and only if the pair $(Y,\Delta+f^*D)$ is slc for every snc divisor $D\subset X$.*

Lemma 6.24. *Let $f\colon (Y,\Delta)\to X$ be a locally stable morphism over a smooth base X, and $\nu\colon \overline{Y}\to Y$ be the normalization of Y with conductor $\overline{C}\subset\overline{Y}$. Then $f\circ\nu\colon (\overline{Y},\overline{\Delta}+\overline{C})\to X$ is locally stable.*

Proof. If $(Y,\Delta+f^*D)$ is slc for any snc divisor $D\subset X$, then $(\overline{Y},\overline{\Delta}+\overline{C}+(f\circ\nu)^*D)$ is lc by [Kol13, Thm. 5.38]. □

Lemma 6.25 ([Pat16, Lem. 2.12])**.** *Let $f\colon (Y,\Delta)\to C$ be a locally stable morphism over an snc curve C. Then (Y,Δ) is slc.*

The moduli part of locally stable lc fibrations admits the following birational characterization.

Lemma 6.26. *Let $f\colon (Y,\Delta)\to X$ be a locally stable lc-trivial fibration inducing the generalized pair $(X,B,\mathbf{M})$. Then $B=0$ and $f^*\mathbf{M}_X\sim_{\mathbb{Q}} K_{Y/X}+\Delta$.*

Proof. Let $(T,0)$ be the spectrum of the local ring of a prime divisor D on any modification of X. Since f is locally stable, the pair (Y_T,Δ_T+Y_0) is lc by [Kol23, (2.3.3)], so $\mathrm{ord}_D(B)=0$ by Theorem 6.12.(2). □

A fibration with K-trivial general fiber can be made locally stable and lc-trivial via an alteration.

Proposition 6.27. *Let $f\colon (Y,\Delta)\to X$ be a fibration of quasiprojective varieties whose general fiber $(Y_{\eta_X},\Delta_{\eta_X})$ is log Calabi–Yau. Then there is a projective, generically finite, dominant morphism $q\colon W^\circ\to X$, and a* *projective compactification $W^\circ \to W$, a locally stable morphism $f'\colon (Y',\Delta') \to W$ such that the pullback of the generic fiber of f along q is crepant birational to the generic fiber of f'.*

Furthermore, we can assume that:

(1) *any closed locus of interest in W is a simple normal crossing divisor;*
(2) *given a polarized variation V of Hodge structure supported on a smooth locally closed subset Z° in X, there exist a proper log smooth scheme (R, D_R) and an embedding $\iota\colon R \hookrightarrow W$ such that the composition $q\circ\iota\colon R\setminus D_R \to Z^\circ$ is projective, generically finite and surjective, and $(q\circ\iota)^*V$ has local unipotent monodromy;*
(3) *$K_{Y'/W} + \Delta' \sim_{\mathbb{Q},f'} 0$; and*
(4) *(Y',Δ') is dlt in codimension 1 over W, i.e., $(Y',\Delta' + f^{-1}(D))$ is dlt over the generic point of any prime divisor D in W.*

Proof. We follow closely [Kol23, Thm. 4.59]. We can replace X with a projective alteration of a compactification of X satisfying (1) and (2). To achieve (2), choose for R an irreducible component, dominating Z°, of a complete intersection of ample divisors in the simple normal crossing divisor $\overline{q^{-1}(Z^\circ)}$. Eventually, first replace X with a projective alteration to grant that the local monodromy of $(q\circ\iota)^*V$ is unipotent.

Now, let $(\widetilde{Y}_{\eta_X}, \widetilde{\Delta}_{\eta_X}) \to (Y_{\eta_X}, \Delta_{\eta_X})$ be a log resolution of the generic fiber of f. By [AK00] (cf., also [ALT20]), there exists a generically finite, dominant map $q\colon W \dashrightarrow X$, from a smooth projective variety W, such that $(\widetilde{Y}_{\eta_X}, \widetilde{\Delta}_{\eta_X}) \times_{\eta_X} \eta_W$ extends to a locally stable morphism $f_1\colon (Y_1,\Delta_1) \to W$ and semistable in codimension 1. By [ALT20, Thm. 4.7], W can be chosen in such a way that (1) and (2) continue to hold. Observe that $(Y_{\eta_X}, \Delta_{\eta_X}) \times_{\eta_X} \eta_W$ extends to a good minimal model $f_2\colon (Y_2,\Delta_2) \to W$; see [HX13, Thm. 1.1]. By [HH20, Thm. 1.7], this ensures that a $(K_{Y_1/W}+\Delta_1)$-MMP with scaling of an ample divisor terminates with a minimal model $f'\colon (Y',\Delta') \to W$, which is again locally stable by [Kol23, Cor. 4.57]. Furthermore, since (Y_1,Δ_1) is semistable in codimension 1, (Y',Δ') is dlt in codimension 1 over W. □

We conclude the section with a technical lemma, used in Theorem 7.2, about the existence of a special sdlt modification of an slc pair over a nodal curve.

Lemma 6.28. *Let $f\colon (Y,\Delta) \to C$ be a locally stable fibration over a connected (strictly) snc curve C with $K_{Y/C} + \Delta \sim_{\mathbb{Q}} 0$. Suppose that (Y,Δ) is an sdlt pair over a dense open set $C^\circ \subset C$. Then there exist a surjective morphism $q'\colon C' \to C$ from a connected (strictly) snc curve and a fibration $f'\colon (Y',\Delta') \to C'$ with the property that*

(1) *(Y',Δ') is an sdlt pair, whose irreducible components each dominate an irreducible component in C';*
(2) *the restriction of f' to any sources dominating an irreducible component of C is an lc-trivial fibration (with connected fibers);*
(3) *$f_*(\omega^{[m]}_{Y'/C'}(m\Delta')) \simeq (q')^*(f_*(\omega^{[m]}_{Y_1/C'}(m\Delta_{C'})))$ for any integer m.*

Proof. We first achieve (2), i.e., the connectedness of the fibers of the sources. Let Z be the closure in Y_C of a stratum of Y_{C°. By [Kol23, Lem. 2.11], the restriction $(f_C)|_Z\colon (Z, \mathrm{Diff}^*_Z(\Delta)) \to C_W$ is a locally stable morphism over an irreducible component C_W of C, but not necessarily a fibration with connected fibers. The finite map $q_W\colon C'_W \to C_W$ in the Stein factorization $W \to C'_W \xrightarrow{q_W} C_W$ cannot be ramified by local stability of $(f_C)|_Z$, so it is étale. Since any étale cover of C_W extends to an étale cover of C, there exists an étale cover $C' \to C$ with the property that any source of $f_{C'}\colon Y_{C'} \to C'$ has connected fibers over the irreducible component that it dominates.

Note that the irreducible components of $Y_{C'}$ are normal in codimension one. Indeed, since (Y,Δ) is sdlt over C°, eventual self-intersections in codimension one lie in fibers over closed points. If the branches of the self-intersection map via $f_{C'}$ to distinct branches of a node in C', then C' is nc not snc, which is a contradiction; otherwise, if the branches of the self-intersection dominate a single branch contained in an irreducible component C_B of C', then a fiber of the locally stable morphism f_{C_B} would be non-reduced, which is a contradiction.

By [Has21], since the irreducible components of $Y_{C'}$ are normal in codimension one, then there exist an sdlt pair (Y', Δ') and a crepant birational morphism $\pi\colon (Y', \Delta') \to (Y_{C'}, \Delta_{C'})$, which is an isomorphism at all codimension 1 singular point of Y' and $Y_{C'}$, such that

$$\pi_*(\omega^{[m]}_{Y'/C'}(m\Delta')) \simeq \omega^{[m]}_{Y_1/C'}(m\Delta_{C'}). \tag{6.5}$$

Taking pushforward along $f_{C'}$ and by [Kol23, (2.67.2)], we achieve (3). □

6.5. **Variation of Hodge structures and source of a degeneration of CY pairs.** The main result of this section is Theorem 6.31: it identifies the transcendental part of the cohomology of the source over a divisor with that of the limiting mixed Hodge structure of a family of (log) Calabi–Yau in the punctured neighbourhood of the divisor. We first define the relevant variations of Hodge structures. To this end, we extend the Construction 6.14 from the generic point of the base X through the generic point of a divisor $D_X \subset X$.

Construction 6.29. Let $f\colon (Y, \Delta) \to X$ be a projective fibration whose generic fiber $(Y_{\eta_X}, \Delta_{\eta_X})$ is klt log Calabi–Yau of dimension n. Fix a smooth integral divisor $D_X \subset X$. Up to shrinking X around the generic point of D_X, by Proposition 6.27, there exist

(1) a projective alteration $q\colon W \to X$;
(2) an lc-trivial locally stable fibration $f'\colon (Y', \Delta') \to W$, locally trivial over $D \coloneqq D_W$ and $W \setminus D$, such that the pullback of the generic fiber of f along q is crepant birational to the generic fiber of f'; and
(3) $(Y', \Delta' + Y'_D)$ is a dlt pair with $\lfloor \Delta' + Y'_D \rfloor = Y'_D$.

Let $g\colon (Y_1, \Delta_1) \to (Y', \Delta' + Y'_D)$ be a log resolution of $(Y', \Delta' + Y'_D)$ with log pullback (Y_1, Δ_1) such that g is an isomorphism over the snc locus of $(Y', \Delta' + Y'_D)$; see [Kol13, Thm. 10.45]. Then, write

$$\Delta_1 \coloneqq E_1 + F_1 - G_1 \quad \text{with} \quad E_1 \coloneqq \Delta_1^{=1}, G_1 \coloneqq \lceil \Delta_1^{<0} \rceil.$$

Since the generic fiber of f is klt, E_1 lies over D. Let d be the minimal positive integer such that dF_1 is an integral divisor and $d(K_{Y_1} + \Delta_1) \sim_{f_1} 0$.[15] Consider $L_1 \coloneqq \mathcal{O}_{Y_1}(G_1 - K_{Y_1} - E_1)$. The isomorphism $L_1^d \simeq \mathcal{O}_{Y_1}(dF_1)$ determines a normalized cyclic cover $a\colon Y_2 \to Y_1$, with Galois group μ_d, branched along F_1; see [KM98, 2.49-53]. Since a is a cyclic cover branched along an snc divisor, Y_2 and all strata of $Y_{2,D}$ have quotient singularities. Let $h\colon (Y_3, (Y_3)_D) \to (Y_2, (Y_2)_D)$ be a μ_d-equivariant log resolution, and set $f_2 \coloneqq f_1 \circ a$ and $f_3 \coloneqq f_2 \circ h$. Generically, the cover a is one of the covers obtained by applying Construction-Definition 6.14 to $f_1\colon Y_1 \to W$; cf. also footnote 15. In particular, f_2 and f_3 are fibrations (with connected fibers). To summarize, we collect the introduced maps over W in the following diagram

$$\begin{tikzcd}
(Y', \Delta') \arrow[d, "f'"'] & (Y_1, \Delta_1) \arrow[l, "g"'] \arrow[dl, "f_1"] & Y_2 \arrow[l, "a"'] \arrow[dll, "f_2"] & Y_3 \arrow[l, "h"'] \arrow[dlll, "f_3"] \\
W & & &
\end{tikzcd}$$

where g is crepant birational, a is a cyclic cover, and h is birational.

Let (S, Δ_S) be a source of $f'\colon (Y'_D, \Delta_{Y'_D}) \to D$. Let S_i be a stratum of $Y_{i,D}$, generically finite onto the source $S \subset Y'$. Generically, the restriction $a\colon S_2 \to S_1$ is again one of the covers obtained by applying

[15] Observe that d is also the minimal positive integer d° such that the previous conditions hold simply along the generic fiber of f_1; a priori d° only divides d. However, by Remark 6.19, we have $d^\circ(K_{Y_1} + \Delta_1) \sim d^\circ f^*(K_W + B_W + \mathbf{M}_W)$, but by (2) the RHS is an integral Cartier divisor, so $d^\circ = d$.

Construction-Definition 6.14 to $f_1 \colon S_1 \to D$. Up to shrinking X again, and replacing W with an étale cover obtained by spreading out the Stein factorization of $f'|_S$, we can suppose that

(4) $f_S \coloneqq f'|_S \colon (S, \Delta_S) \to D_W$ is an lc-trivial fibration (in particular with connected fibers) of relative dimension m.

By Construction-Definition 6.14, the moduli part of the lc-trivial fibration $f' \colon (Y', \Delta') \to W$ is the Hodge bundle of the χ-isotypic component of the Deligne/Schmid extension of $R^n(f_2)_*\mathbb{C}$ (or equivalently of $R^n(f_3)_*\mathbb{C}$ as in Remark 6.18), i.e.,

$$\mathbf{M}_W \coloneqq F^n(\mathcal{R}^n(f_2)_*\mathbb{C}_{Y_2^\circ})^{\mathrm{tr}}_\chi \simeq F^n(\mathcal{R}^n(f_3)_*\mathbb{C}_{Y_3^\circ})^{\mathrm{tr}}_\chi \simeq ((f_3)_*\omega_{Y_3/W})_\chi.$$

Notation 6.30. Denote by V_Y and V_S the Deligne/Schmid extension of the μ_d-equivariant polarizable variations of pure Hodge structures $R^n(f_3)_*\mathbb{C}_{Y_3^\circ}$ and $R^m(f_3)_*\mathbb{C}_{S_3^\circ}$.

Theorem 6.31. *In the notation above, there exists an isomorphism of variations of Hodge structures*

$$(V_Y|_D)^{\mathrm{tr}}_\chi \simeq (V_S)^{\mathrm{tr}}_\chi.$$

Proof. The required isomorphism is obtained by composing the isomorphisms (6.6), (6.8), and (6.11).

Step 1. Since V_Y is a CY variation of Hodge structure, there exists a unique integer k such that $\mathrm{gr}^n_F \mathrm{gr}^W_{n+k}(V_{Y,\mathcal{O}})_\chi \neq 0$. Recall that the logarithmic monodromy N defining the weight filtration has type $(-1,-1)$. Then the isomorphism $N^k \colon \mathrm{gr}^W_{n+k}(V_Y|_D)_\chi \to \mathrm{gr}^W_{n-k}(V_Y|_D)_\chi$ induces an isomorphism

$$F^n(V_{Y,\mathcal{O}}|_D)_\chi \simeq (\mathrm{gr}^W_{n+k}(V_{Y,\mathcal{O}}|_D)_\chi)^{n,k} \xrightarrow[N^k]{\simeq} (\mathrm{gr}^W_{n-k}(V_{Y,\mathcal{O}}|_D)_\chi)^{n-k,0} \xrightarrow[\text{conj.}]{\simeq} \overline{(\mathrm{gr}^W_{n-k}(V_{Y,\mathcal{O}}|_D)_\chi)^{0,n-k}} \simeq \overline{\mathrm{gr}^0_F(V_{Y,\mathcal{O}}|_D)_\chi}.$$

In particular, $N^k(V_Y|_D)^{\mathrm{tr}}_\chi$ is the unique minimal subvariation of Hodge structure in $\mathrm{gr}^W_{n-k}(V_Y|_D)_\chi$ containing $\mathrm{gr}^0_F(V_{Y,\mathcal{O}}|_D)_\chi$, denoted $(V_Y|_D)^{\overline{\mathrm{tr}}}_\chi$, i.e.,

$$(V_Y|_D)^{\mathrm{tr}}_\chi \simeq (V_Y|_D)^{\overline{\mathrm{tr}}}_\chi. \tag{6.6}$$

Step 2. Let ψ_D be the vanishing cycle functor associated to a global function defining $Y_{3,D}$, which exists up to further shrinking W. The specialization morphism $\mathbb{Q}_{Y_{3,D}} \to \psi_D\mathbb{Q}_{Y_3}$ between Hodge modules on $Y_{3,D}$ (cf. [Sai90, (2.24.3)] or [PS08, Thm. 11.29]) induces a μ_d-equivariant morphism of polarizable variation of mixed Hodge structures

$$R^n(f_3)_*\mathbb{Q}_{Y_{3,D}} \to R^n\psi_D\mathbb{Q}_{Y_3} \simeq V_Y|_D, \tag{6.7}$$

where the last isomorphism follows, e.g., by the remark right before [Ste75, (5.11) Cor.]. The morphism (6.7) induces an isomorphism

$$\mathrm{gr}^0_F R^n(f_3)_*\mathbb{C}_{Y_{3,D}} \simeq \mathrm{gr}^0_F(V_{Y,\mathcal{O}}|_D)_\chi,$$

which entails the isomorphism of variation of pure Hodge structures

$$(V_Y|_D)^{\overline{\mathrm{tr}}} \simeq (R^n(f_3)_*\mathbb{Q}_{Y_{3,D}})^{\overline{\mathrm{tr}}}_\chi \simeq (R^n(f_2)_*\mathbb{Q}_{Y_{2,D}})^{\overline{\mathrm{tr}}}_\chi; \tag{6.8}$$

see [KLS21, Cor. 5.7, Proof of Prop. 9.2].

Step 3. Write $Y_{2,D} \coloneqq R = \bigcup_{i\in I} R_i$ as a union of its irreducible components, and fix an ordering of I. Denote by $R^{[k]}$ the disjoint union of the strata that have codimension k in R. The Mayer–Vietoris complex of $\mathbb{Q}_R$ associated to the closed cover $\{R_i\}_{i\in I}$

$$\mathbb{Q}_{R^\bullet} \colon \mathbb{Q}_{R^{[0]}} \to \mathbb{Q}_{R^{[1]}} \to \mathbb{Q}_{R^{[2]}} \to \dots \tag{6.9}$$

is a μ_d-equivariant resolution of $\mathbb{Q}_R$. Recall that the differential of the complex is induced by the natural restriction $\mathbb{Q}_{R_J} \to \mathbb{Q}_{R_{J\cup\{j\}}}$, with a plus or a minus sign according to the parity of the position of j in $J\cup\{j\}$; cf. [KLSV18, App. A]. There exists a μ_d-equivariant spectral sequence

$$E_1^{p,q} = R^p(f_2)_*\mathbb{Q}_{R^{[q]}} \implies R^{p+q}(f_2)_*\mathbb{Q}_R. \tag{6.10}$$

Since the differentials of the spectral sequence are morphisms of variations of mixed Hodge structures and all strata of R are proper with quotient singularities (hence they are rational homology manifolds), the spectral sequence (6.10) induces the μ_d-equivariant spectral sequence

$$\mathrm{gr}_F^0 E_1^{p,q} = \mathrm{gr}_F^0 R^p(f_2)_*\mathbb{Q}_{R^{[q]}} = \mathrm{gr}_F^0 \mathrm{gr}_p^W R^p(f_2)_*\mathbb{C}_{R^{[q]}} \implies \mathrm{gr}_F^0 R^{p+q}(f_2)_*\mathbb{C}_R,$$

abutting at the E_2 page for weight reasons. Together with Lemma 6.32, we obtain

$$\mathrm{gr}_F^0(R^n(f_2)_*\mathbb{C}_R)_\chi \simeq \mathrm{gr}_F^0 E_{\infty,\chi}^n \simeq \mathrm{gr}_F^0 E_{2,\chi}^n \twoheadrightarrow \mathrm{gr}_F^0 E_{2,\chi}^{m,n-m} \twoheadrightarrow \mathrm{gr}_F^0(R^m(f_2)_*\mathbb{C}_{S_2})_\chi.$$

Since both $\mathrm{gr}_F^0(R^n(f_2)_*\mathbb{C}_R)_\chi$ and $\mathrm{gr}_F^0(R^m(f_2)_*\mathbb{C}_{S_2})_\chi$ are line bundles, the last two surjective maps are actual isomorphisms, and so

$$\mathrm{gr}_F^0(R^n(f_2)_*\mathbb{C}_R)_\chi \simeq \mathrm{gr}_F^0(R^m(f_2)_*\mathbb{C}_{S_2})_\chi$$

(as an aside, this also shows that $k = m$). Therefore, we conclude

$$(R^n(f_2)_*\mathbb{Q}_{Y_{2,D}})_\chi^{\overline{\mathrm{tr}}} \simeq (R^m(f_2)_*\mathbb{Q}_{S_2})_\chi^{\overline{\mathrm{tr}}} = (V_S)_\chi^{\mathrm{tr}}, \tag{6.11}$$

where the last equality follows again from the fact that S_2 has quotient singularities. □

We prove the lemma used in the proof of Theorem 6.31. Recall that n (resp. m) is the relative dimension of the morphisms $Y_i \to W$ (resp. $S_i \to D$).

Lemma 6.32. *In the notation of Step 3 of the proof of Theorem 6.31, we have*

$$\mathrm{coker}(d_1\colon\ \mathrm{gr}_F^0 E_{1,\chi}^{m,n-m-1} \to \mathrm{gr}_F^0 E_{1,\chi}^{m,n-m}) \twoheadrightarrow \mathrm{gr}_F^0(R^m(f_2)_*\mathbb{C}_{S_2})_\chi.$$

Proof. Since all strata of R have quotient (hence du Bois) singularities, we write

$$\mathrm{gr}_F^0 E_{1,\chi}^{p,q} \simeq (R^p(f_2)_*\mathcal{O}_{R^{[q]}})_\chi \simeq R^p(f_1)_*\mathcal{O}_{a(R^{[q]})}(-L_1).$$

Write $a(R^{[q]}) = LCC^{[q]} \cup a(R^{[q]})'$, where $LCC^{[q]}$ is the disjoint union of the lc centers of (Y_1, Δ_1) of dimension $n-q$, and $a(R^{[q]})'$ is the disjoint union of the residual $(n-q)$-dimensional strata of $Y_{1,D}$ that are not lc centers. For brevity, set $A^{p,q} \coloneqq R^p(f_1)_*\mathcal{O}_{LCC^{[q]}}(-L_1)$ and $B^{p,q} \coloneqq R^p(f_1)_*\mathcal{O}_{a(R^{[q]})'}(-L_1)$. Since the only strata of $a(R^{[n-m-1]})$ containing a minimal lc center are lc centers, the differential

$$d_1\colon A^{m,n-m-1} \oplus B^{m,n-m-1} \to A^{m,n-m} \oplus B^{m,n-m}$$

is lower triangular; cf. (6.9). We determine the upper block of

$$d_A \coloneqq pr_{A^{m,n-m}} \circ d_1 \circ i_{A^{m,n-m-1}} \circ\colon A^{m,n-m-1} \to A^{m,n-m},$$

where pr and i denote the natural projections and inclusions.

To this end, let $Z_1 \subset Y_1$ be an irreducible component of $LCC^{[n-m-1]}$. The trace of E_1 on Z_1, denoted by E_{Z_1}, is the restriction to Z_1 of the components of E_1 not containing Z_1. By Construction 6.29, the log resolution $g\colon Y_1 \to Y'$ is an isomorphism over the snc locus of the dlt pair $(Y', \Delta' + Y_D')$, hence at the generic point of Z_1. By [Kol13, (4.6),(4.7.1), Thm. 4.19], the induced map

$$g\colon (Z_1, E_{Z_1} + (F_1 - G_1)|_{Z_1}) \to (Z', \mathrm{Diff}_{Z'}^*(\Delta' + Y_D')) \coloneqq (g(Z_1), g_*(E_{Z_1} + (F_1 - G_1)|_{Z_1}))$$

is crepant birational, and $(Z', \mathrm{Diff}_{Z'}^*(\Delta' + Y_D'))$ is an (effective) dlt pair. In particular, g must contract $G_1|_{Z_1}$ and maps E_{Z_1} birationally onto $E_{Z'} \coloneqq \mathrm{Diff}_{Z'}^*(\Delta' + Y_D')^{=1}$. We obtain

$$Rg_*\mathcal{O}_{Z_1}(G_1|_{Z_1} - E_{Z_1}) \simeq g_*\mathcal{O}_{Z_1}(G_1|_{Z_1} - E_{Z_1}) \simeq \mathcal{O}_{Z'}(-E_{Z'}),$$

where the first isomorphism follows from [Kol13, Cor. 10.38.(1)] since $G_1|_{Z_1} - E_{Z_1} \sim_{\mathbb{Q},g} K_{Z_1} + F_1|_{Z_1}$, and the second isomorphism follows by the normality of Z' and Z_1 and since $G_1|_{Z_1} - E_{Z_1} + g^*E_{Z'}$ is effective and g-exceptional. Pushing forward along f_1 and using relative duality, we obtain

$$(R^m(f_1)_*\mathcal{O}_{Z_1}(-L_1))^\vee \simeq R^1(f_1)_*(\omega_{Z_1/D}\otimes L_1) \simeq (R^1(f_1)_*\mathcal{O}_{Z_1}(G_1|_{Z_1} - E_{Z_1}))\otimes\omega_D^{-1} \simeq (R^1 f'_*\mathcal{O}_{Z'}(-E_{Z'}))\otimes\omega_D^{-1}.$$

Note that

- $E_{Z'}$ has either one or two irreducible components by [Kol13, Prop. 4.37]; and
- $R^1 f'_*\mathcal{O}_{Z'} \hookrightarrow R^1 f'_*\mathcal{O}_{E_{Z'}}$ by [BFPT24, Lem. 3.2].

Pushing forward along f' the short exact sequence $0 \to \mathcal{O}_{Z'}(-E_{Z'}) \to \mathcal{O}_{Z'} \to \mathcal{O}_{E_{Z'}} \to 0$, we obtain

$$R^1 f'_*\mathcal{O}_{Z'}(-E_{Z'}) \simeq \begin{cases} R^0 f'_*\mathcal{O}_S & \text{if } E_{Z'} = S \sqcup S', \\ 0 & \text{if } E_{Z'} \text{ is connected.} \end{cases}$$

To summarize, if Z_1 is an lc center containing two distinct minimal lc centers S_1 and S'_1, then

$$R^m(f_1)_*\mathcal{O}_{Z_1}(-L_1) \simeq R^m(f_1)_*\mathcal{O}_{S_1}(-L_1); \tag{6.12}$$

otherwise $R^m(f_1)_*\mathcal{O}_{Z_1}(-L_1) \simeq 0$.

Let Γ be the (oriented) graph whose vertices V_Γ are the minimal lc centers of (Y_1, Δ_1), and whose edges are lc centers of dimension $m+1$ joining two minimal lc centers. Then by definition of d_1 and (6.12), the map d_A can be identified with $\delta \otimes \mathrm{id}_{R^m(f_1)_*\mathcal{O}_{S_1}(-L_1)}$, where $\delta\colon \mathbb{C}[E_\Gamma] \to \mathbb{C}[V_\Gamma]$, $\delta(e) = e_0 - e_1$ is the boundary map of the graph Γ. By [Kol13, Thm. 4.40], Γ is connected, so

$$\begin{aligned} \operatorname{coker}(d_1^{LCC}) = \operatorname{coker}(\delta) \otimes R^m(f_1)_*\mathcal{O}_{S_1}(-L_1) &\simeq H^0(\Gamma, \mathbb{C}) \otimes R^m(f_1)_*\mathcal{O}_{S_1}(-L_1) \\ &\simeq R^m(f_1)_*\mathcal{O}_{S_1}(-L_1) \simeq \mathrm{gr}_F^0(R^m(f_2)_*\mathbb{C}_{S_2})_\chi. \end{aligned}$$

□

7. B-semiampleness conjecture

7.1. Proof of the b-semiampleness conjecture.

7.1.1. *Proof of Theorem 1.5.* Let $f\colon (Y, \Delta) \to X$ be an lc-trivial fibration inducing the generalized pair $(X, B_X, \mathbf{M})$. Up to taking a modification of X and the corresponding normalized fiber product of Y, we may assume that all varieties involved are quasiprojective. The b-semiampleness conjecture for lc (or slc) generic fiber is equivalent to the b-semiampleness conjecture for klt generic fiber, via subadjunction to a source; see [FG14b, Thm. 1.1] or Remark 7.4. Moreover, the statement of the conjecture is insensitive to alteration of the base; cf. Remark 6.16. Therefore, we can suppose:

(‡) (Y, Δ) is a klt quasiprojective pair over the generic point of X, and that properties (†) and (††) in Construction-Definition 6.14 hold.

Then Construction-Definition 6.14 and Remark 6.17 imply the existence of:

1. a snc pair (X, D);
2. a polarizable integral variation of pure Hodge structures on $X \setminus D$ whose Deligne/Schmid extension on X is denoted V_Y;
3. a complex CY variation of Hodge structures $(V_Y)^{\mathrm{tr}}_\chi \subseteq V_{Y,\mathbb{C}}$ whose Hodge bundle is $\mathbf{M}_X$;
4. a polarizable integral pure CY variation V_{CY} of Hodge structures on $X \setminus D$ with unipotent local monodromy whose Hodge bundle is $\mathbf{M}_X$.

In view of Theorem 4.1, to prove the b-semiampleness conjecture, it suffices to show that the moduli part is integrable and has torsion combinatorial monodromy. This is the content of Theorem 7.1 and Theorem 7.2.

Theorem 7.1 (Integrability of $\mathbf{M}$)**.** *Under the assumption (‡), the moduli part $\mathbf{M}_X$ is integrable.*

Proof. It suffices to show that for any integral subvariety $Z \subset \overline{X_\Sigma}$ such that the restriction of $\mathbf{M}_X$ is not big, the period map of $(V_Y|_Z)^{\mathrm{tr}}_\chi$ is not generically immersive.

We first alter X, Y, Z in order to compare the relevant variations of Hodge structure and set the inductive argument on the dimension of the source. By Proposition 6.27, there exist

(1) a projective alteration $q_1 \colon W_1 \to X$,
(2) an lc-trivial locally stable fibration $f_1 \colon (Y_1, \Delta_1) \to W_1$, dlt in codimension one, such that the pullback of the generic fiber of f along q_1 is crepant birational to the generic fiber of f_1.
(3) a prime divisor $E \subset W_1$ dominating Z;
(4) a proper snc pair (R, D_R), where R is a general complete intersection in E, mapping generically finite onto Z,

such that $(q_1^* V_Y)^{\mathrm{tr}}_{\chi}$ has local unipotent monodromy on $R \setminus D_R$. By §6.5, up to a further finite dominant morphism $q_2 \colon W_2 \to R$, a source of $(f_1)_{W_2} \colon ((Y_1)_{W_2}, (\Delta_1)_{W_2}) \to W_2$, denoted $f_S \colon (S, \Delta_S) \to W_2$, is an lc-trivial fibration

(1) inducing the generalized pair $(W_2, B_{W_2}, \mathbf{N})$,
(2) satisfying properties (†) and (††) (in particular $\mathbf{N}$ descends on W_2), and
(3) such that

$$(q^* V_Y)^{\mathrm{tr}}_{\chi} \simeq (V_S)^{\mathrm{tr}}_{\chi} \quad \text{and} \quad q^* \mathbf{M}_X \simeq \mathbf{N}_{W_2}, \tag{7.1}$$

where $q \coloneqq q_1 \circ q_2 \colon W_2 \to X$. Note also that f_S is locally stable by Lemmas 6.22, 6.24 and [Kol23, cf. proof of Lem. 2.11], so $f_S^* \mathbf{N}_{W_2} \sim_{\mathbb{Q}} K_{S/W_2} + \Delta_S$ by Lemma 6.26.

Now, assume that $\mathbf{M}_X|_Z$ is not big. Then $\mathbf{N}_{W_2}$ is so too. By [Amb05, Thm. 3.3] or [PZ20, Thm. A.12], there exist curves passing through the general point of W_2 over which f_S is isotrivial, so the period map of $(V_S)^{\mathrm{tr}}_{\chi}$ (resp. of V_S^{tr} and $V_{S,CY}^{\mathrm{tr}}$) is not generically immersive. By (7.1), we conclude that the period map of $(V_Y|_Z)^{\mathrm{tr}}_{\chi}$ (resp. $(V_{CY}|_Z)^{\mathrm{tr}}_{\chi}$) is not generically immersive. □

Theorem 7.2 (Torsion combinatorial monodromy of $\mathbf{M}$)**.** *Under the assumption (‡), the moduli part $\mathbf{M}_X$ has torsion combinatorial monodromy.*

Proof. Let C be a proper connected strictly nodal curve with normalization ν_C, and $q \colon C \to X$ be a morphism from a proper strictly nodal curve C such that $(q \circ \nu_C)^* \mathbf{M}_X$ is trivial. We show that the canonical flat connection on $q^* \mathbf{M}_X$ has torsion monodromy. Observe that, in the statement, we can always replace C with proper connected strictly nodal curves dominating C.

Step 1. We first construct an sdlt modification of f over C. By Proposition 6.27, there exist

(1) a projective alteration $q_1 \colon W \to X$,
(2) an lc-trivial locally stable fibration $f_1 \colon (Y_1, \Delta_1) \to W$, dlt in codimension one, such that the pullback of the generic fiber of f along q_1 is crepant birational to the generic fiber of f_1.

such that

$$q_1^* \mathbf{M}(f) = \mathbf{M}(f_1) \qquad \text{and} \qquad \mathbf{M}_W \coloneqq \mathbf{M}(f_1)_W \sim_{\mathbb{Q}} \frac{1}{k} (f_1)_* (\omega^{[k]}_{Y_1/W}(k\Delta_1)),$$

where k is a sufficiently divisible positive integer, and $q_1^{-1}(C)$ has simple normal crossings. Replace C with a (connected) proper strictly nodal complete intersection in $q_1^{-1}(C)$ dominating C. In particular, we can suppose that the locally stable fibration $f_{1,C} \colon Y_{1,C} \to C$ is sdlt over a dense open set $C^\circ \subset C$. By Lemma 6.28, there exist a morphism $q' \colon C' \to C$ from a connected stricly snc curves and an lc fibration $f' \colon (Y', \Delta' \coloneqq \Delta_{1,C'}) \to C'$ with the property that

(1) (Y', Δ') is an sdlt pair, whose irreducible components each dominate an irreducible component in C';
(2) the restriction of f' to any sources dominating an irreducible component of C' is an lc-trivial fibration (with connected fibers);
(3) $f'_* \omega^{[k]}_{Y'/C'}(k\Delta') \simeq k(q \circ q')^* \mathbf{M}_X$, where k is a sufficiently divisible positive integer.

Step 2. We are left to show that if $f'_*\omega^{[k]}_{Y'/C'}(k\Delta')$ is trivial on each component of C', then the monodromy of $f'_*\omega^{[k]}_{Y'/C'}(k\Delta')$ is torsion. To this end, observe that any source $f_S\colon S \to C'_S$, for some irreducible component of C', is an lc-trivial fibration with $\mathbf{M} \sim_{\mathbb{Q}} 0$, since $\mathbf{M}_{C'} \sim_{\mathbb{Q}} (q \circ q')^*\mathbf{M}_X$ is trivial by assumption along the irreducible components of C'. Hence, by the compatibility of locally stable families with base change, [Amb05, Thm. 3.3 and Prop. 4.4] or [PZ20, Thm. A.12], f_S is an isotrivial families, i.e., all fibers are isomorphic to one another, which allows to identify minimal lc centers over adjacent nodes of C'.

The intersection complex of an sdlt variety is the dual polyhedron of its dual complex. Denote by $\Delta(C')$ and $\Delta(Y')$ the intersection complex of the snc curve C' and that of the sdlt variety Y'. The 1-skeleton of $\Delta(Y')$, denoted $\Delta(Y')_1$, consists of two types of edges:

- (dominating edge) those corresponding to sources of Y' dominating a component of C', which are isotrivial families of fiberwise minimal centers; and
- (edge of $\mathbb{P}^1$-link type) those corresponding to $\mathbb{P}^1$-link between minimal strata of Y', mapping to a node of C'.

Note that $\Delta(Y')_1$ is connected by the isotriviality of sources over dominating edges and [Kol13, Thm. 4.40] for edges of $\mathbb{P}^1$-link type. Also, by construction there is a surjective simplicial map $\Delta(Y')_1 \to \Delta(C')_1 = \Delta(C')$. In particular, each loop in $\Delta(C')$ admits a (non-unique) lift γ in $\Delta(Y')_1$. Fix it once for all. Recall that any vertex of γ corresponds to the source W. A dominating edge e with vertices s, t induces an isomorphism of minimal lc centers $(Z_s, \Delta_{Z_s}) \to (Z_t, \Delta_{Z_t})$ by the isotriviality of the family of sources. An edge of $\mathbb{P}^1$-link type with vertices s, t induces a crepant birational map $(Z_s, \Delta_{Z_s}) \dashrightarrow (Z_t, \Delta_{Z_t})$ by [Kol13, Thm. 4.40]. Following the birational identification along the loop γ, we obtain a birational automorphism of a fixed reference minimal lc center (Z, Δ_Z) of a general fiber, which induces a representation $\mathbb{Z}\gamma \to \mathrm{Aut}(H^0(Z, \omega_Z^{[2k]}(2k\Delta_Z)))$, which is trivial for k large and divisible enough by the finiteness of B-representations [HX16, Thm. 1.2]. ☐

This conclude the proof of Theorem 1.5. ☐

7.1.2. *B-semiampleness for projective morphisms between complex analytic spaces.* In recent years, there has been quite some activity in extending the usual MMP to the context of Kähler spaces or analytic varieties. The b-semiampleness of the moduli part for projective morphisms of complex analytic spaces can be reduced to the algebraic case of Theorem 1.5 as follows. In particular, we drop the quasiprojectivity assumption in (‡).

Theorem 7.3. *Let (Y, Δ) be a normal complex analytic space with a sub-pair structure, and let $f\colon (Y, \Delta) \to X$ be a projective fibration between complex analytic spaces. Assume that $K_Y + \Delta \sim_{\mathbb{Q}, f} 0$ and that the general fiber of f is an lc pair. Then, the moduli part of f is b-semiample.*

Proof. Without loss of generality, we can assume that the generic fiber is klt by Remark 7.4. Let $F\colon (\mathcal{Y}, \Xi) \to \mathcal{X}$ be a Hilbert scheme parametrizing general fibers of the projective morphism $f\colon (Y, \Delta) \to X$. Up to replacing X with a modification, we can suppose that the classifying map $\Psi\colon X \dashrightarrow \mathcal{X}$ is an analytic morphism, by applying Hironaka's flattening theorem [Hir75, Cor. 1] to f and the components of $\mathrm{Supp}(\Delta)$ dominating X. Replacing $\mathcal{X}$ with the Zariski closure of the image of $\Psi\colon X \to \mathcal{X}$, we can attribute coefficients to the irreducible components of Ξ such that $(Y_U, \Delta_U) \simeq (\mathcal{Y}, \Xi) \times_{\mathcal{X}} U$, over a dense open set $U \subset X$. In particular, there exists a dense open subset $\mathcal{U} \subset \mathcal{Z}$ such that $(\mathcal{X}_{\mathcal{U}}, \Xi_{\mathcal{U}})$ is a pair ([HX15, Prop. 2.4]) with klt singularities, and $(\mathcal{X}_{\mathcal{U}}, \Xi_{\mathcal{U}}) \to \mathcal{U}$ is an lc-trivial fibration.

Up to an alteration, the moduli part of f is the Hodge bundle of a variation of Hodge structures obtained by the algorithm in Construction-Definition 6.14; see Remark 6.20. Construction-Definition 6.14 applied to $F_{\mathcal{U}}$ pulls back to the analogous construction for f_U, up to eventually shrinking U. Replacing X and $\mathcal{X}$ with compatible alterations, there exists a morphism $\Psi\colon X \to \mathcal{X}$, and polarizable variations of Hodge structures V_U and $V_{\mathcal{U}}$ on dense open subsets $U \subset X$ and $\mathcal{U} \subset \mathcal{X}$, whose complements are snc divisors, and such that:

(1) $\Psi|_U^* V_{\mathcal{U}} = V_U$, and (2) the Schmid extension of the deepest piece of the Hodge filtration of V_U (resp. $V_{\mathcal{U}}$) is the moduli part. By the functoriality of the Schmid's extension, we have $\Psi^* \mathbf{M}_{\mathcal{X}} = \mathbf{M}_X$. Hence, the semiampleness of the moduli part of f in the analytic case follows from the semiampleness of the moduli part of $F_{\mathcal{U}}$ in the algebraic case, proved in Theorem 1.5. ☐

Remark 7.4. Let $f\colon (Y,\Delta) \to X$ be a morphism as in Theorem 7.3 and $f_S\colon (S,\Delta_S) \to X$ be a source of (Y,Δ). Note that $\mathbf{M}(f) \sim_{\mathbb{Q}} \mathbf{M}(f_S)$ as in (6.11). This means that the moduli part of an lc-trivial fibration from an lc pair is the moduli part of an lc-trivial fibration from a klt pair. This reproves [FG14b, Thm. 1.1] without using semistable reduction, in particular it holds over an analytic base as well.

7.2. **Applications of the b-semiampleness conjecture and open questions.** In this section, we collect some immediate applications of the b-semiampleness of the moduli part of an lc-trivial fibration.

7.2.1. *Image of lc pairs.* We show that the image of an lc pair under an lc-trivial fibration is again lc, generalizing [Amb05, Thm. 4.1] and [FG14b, Lem. 1.1].

Theorem 7.5. *Let (Y,Δ_Y) be an lc (resp. klt) pair and $f\colon (Y,\Delta_Y) \to X$ be a projective fibration with $K_Y + \Delta_Y \sim_{\mathbb{Q}} 0$. Then there exists an lc (resp. klt) pair (X,Δ_X) such that $K_Y + \Delta_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta_X)$.*

Proof. The moduli part of the generalized lc pair $(X, B_X, \mathbf{M})$ induced by f (cf. Theorem 6.12) is b-semiample by Theorem 1.5. Apply then [EFG+25, Lem. 4.3]. ☐

We expect that Theorem 7.5 could be a key ingredient in inductive arguments in birational geometry. In the klt case, a version of Theorem 7.5 for klt sub-pairs is used to reduce the finite generation of canonical rings to the general type case; see [BCHM10, Cor. 1.1.2]. According to [BGLM24, §1.8], Theorem 7.5 was one of the missing ingredients to prove that reductive quotients of lc pairs are again lc.

7.2.2. *Adjunction and inversion of adjunction.* The lc centers of a dlt pair (X,Δ) coincide with the irreducible components of the strata of $\Delta^{=1}$. In particular, adjunction to an lc center of any dimension can be performed using the usual residue theory iteratively, as with prime divisors; see [Kol13, §4.2]. In general, to induce a structure of a pair on an lc center Z of an arbitrary lc (not dlt) pair (X,Δ) is more complicated. Roughly speaking, one performs the following steps:

(1) take a dlt modification $(X',\Delta') \to (X,\Delta)$;
(2) choose a prime divisor S in $\mathrm{Supp}(\Delta')^{=1}$ dominating Z;
(3) perform dlt adjunction of (X',Δ') to S, thus obtaining a pair (S,Δ_S);
(4) consider the lc-trivial fibration $(S,\Delta_S) \to W$, where W denotes the Stein factorization of $S \to Z$;
(5) utilize the canonical bundle formula to induce a pair structure on W; and
(6) descend this latter structure to the normalization Z^ν of Z.

For more details, we refer to [FG12, §4]. For this reason, so far, it was only possible to induce the structure of klt pair on minimal lc centers of lc pairs. Thanks to Theorem 1.5, we can generalize this construction to any lc center.

Theorem 7.6 (Adjunction and inversion of adjunction)**.** *Let (X,Δ) be a pair and Z be an lc center. Then, the normalization Z^ν can be endowed with a pair structure (Z^ν, Δ_{Z^ν}) with the following properties:*

(1) $K_{Z^\nu} + \Delta_\nu \sim_{\mathbb{Q}} (K_X + \Delta)|_{Z^\nu}$; and
(2) (X,Δ) is lc in a neighborhood of Z if and only if (Z^ν, Δ_{Z^ν}) is lc.

Proof. We apply the construction in [FG12, §4]. Thus, (1) holds by construction. Then, by Theorem 7.5 and the construction adopted, the "only if" part of (2) follows. Thus, we are left with showing that, if (Z^ν, Δ_{Z^ν}) is lc, then so is (X,Δ) in a neighborhood of Z. For general lc centers, due to the lack of Theorem 1.5, adjunction could only be formulated by using b-divisors on Z^ν; see [Hac14, FH22]. The approaches in

[Hac14, FH22] are proved equivalent in [FH23]. The b-divisor considered in [Hac14] is exactly the boundary b-divisor of the canonical bundle formula considered in [FG12, §4]. In particular, in [Hac14], inversion of adjunction is formulated by requiring that the boundary b-divisor has coefficient at most 1 on any model. By Theorem 7.5 and the construction adopted, this condition is equivalent to requiring that the pair (Z^ν, Δ_{Z^ν}) is lc. Then, the claim follows. $\square$

7.2.3. *Comment about boundary with $\mathbb{R}$-coefficients.* Throughout this work, for a pair (X, Δ) we assume that Δ has coefficients in $\mathbb{Q}$. For many applications, it is important to extend results to pairs with coefficients in $\mathbb{R}$. We remark that Theorem 7.5 and Theorem 7.6 also hold for pairs with real coefficients, provided that all linear equivalences are taken to be over $\mathbb{R}$. This relies on the approximation of pairs with real coefficients by means of convex combinations of pairs with rational coefficients; see, e.g., [HL21].

Indeed, let $f\colon (Y, \Delta) \to X$ be an lc-trivial fibration where (X, Δ) is a quasi-projective $\mathbb{R}$-pair. By [HL21, Lem. 4.1], we may write the $\mathbb{R}$-divisor $K_Y + \Delta = \sum c_i(K_Y + \Delta_i)$ as a real convex combination of finitely many $\mathbb{Q}$-Cartier divisors $K_Y + \Delta_i$ such that: (1) the non-lc and non-klt loci of the pairs (Y, Δ_i) agree with the corresponding ones of (Y, Δ) for all i; and (2) $K_Y + \Delta_i \sim_{\mathbb{Q},f} 0$ for all i. Let (X_i, B_i) be the lc $\mathbb{Q}$-pair obtained applying Theorem 7.5 to the lc-trivial fibration $f\colon (Y, \Delta_i) \to X$. To deduce Theorem 7.5 for (Y, Δ), take the lc $\mathbb{R}$-pair $(X, \sum c_i B_i)$. Moreover, to obtain the corresponding version of Theorem 7.6 for (Y, Δ), we utilize the pair just constructed whenever the canonical bundle formula is invoked in the proof, together with inversion of adjunction for fiber spaces and the fact that (Y, Δ) and (Y, Δ_i) have the same classes of singularities.

7.2.4. *Effective b-semiampleness.* Let $f\colon (Y, \Delta) \to X$ be an lc-trivial fibration inducing the generalized pair $(X, B_X, \mathbf{M})$. The effective b-semiampleness conjecture predicts the existence of a universal positive integer c, only depending on the relative dimension of f and the coefficients of the horizontal part of Δ, such that $c\mathbf{M}$ is b-free, or eventually weaker statements involving other topological invariants of the general fiber. The conjecture was formulated by Prokhorov and Shokurov [PS09, Conj. 7.13.3], and proved in *loc. cit.* in relative dimension 1. Recently, the conjecture has also been solved for lc-trivial fibrations whose general fiber is an abelian or primitive symplectic variety with bounded second Betti number of fixed dimension; see [EFG+25, Thm. C]. This was a key step towards the proof of boundedness results for certain K-trivial fibrations in [EFG+25]. In particular, combining [EFG+25] and [ABB+23, Thm. 1.4], the conjecture is also settled in relative dimension 2.

It is natural to ask whether a fixed positive power of the Schmid extension of the Griffiths bundles of a polarizable integral variation of Hodge structures sharing the same period domain is free. Analogously, whether the same holds for the Hodge bundle of integrable polarizable CY variations of Hodge structure with torsion combinatorial monodromy sharing the same period domain. A positive answer to this question may entail boundedness results for more general K-trivial fibrations.

While the effective version of the b-semiampleness conjecture remains open at the moment, Corollary 7.8 below provides a tool to prove it when the general fibers belong to a given bounded family of pairs. Thus, we deduce the effective version of the conjecture when the general fiber is a klt log CY pair of Fano type. In particular, we obtain the following application, which was kindly pointed out to us by Shokurov.

Corollary 7.7. *Let $f\colon (Y, \Delta) \to X$ be an lc-trivial fibration of relative dimension n from a pair (Y, Δ) such that Δ is effective over the generic point of X. Further assume the following:*

(i) the generic fiber of f is a klt pair;
(ii) Δ is big over X.

Let $\mathbf{M}$ denote the moduli b-divisor induced by f. Then, there is a constant I, only depending on n and the horizontal multiplicities of Δ, such that $I\mathbf{M}$ is b-Cartier and b-free.

Proof. Since the moduli b-divisor is determined by the general behavior of the fibration, we may shrink X so that (Y, Δ) is a klt pair and Δ has no f-vertical components. Then, we may replace (Y, Δ) with a small $\mathbb{Q}$-factorialization. In particular, we may assume that Δ is a $\mathbb{Q}$-Cartier divisor that is f-big. Notice that $(Y, (1+\epsilon)\Delta)$ is klt for $0 < \epsilon \ll 1$ and $K_Y + (1+\epsilon)\Delta$ is f-big.

Thus, we may apply [BCHM10] to $(Y, (1+\epsilon)\Delta)$ and the morphism f and replace Y with the relatively ample model of Δ. In particular, Y may no longer be $\mathbb{Q}$-factorial, but we gain the fact that Δ is f-ample. Notice that all these operations do not affect $\mathbf{M}$.

Up to shrinking X, we may assume that all the fibers (Y_x, Δ_x) of f are klt pairs. Then, these belong to a bounded family of pairs by [HX15]. In particular, there is a constant C, independent of f or of the point $x \in X$ (recall X has been shrunk), such that $C\Delta_x$ is ample and Cartier. In order to construct a relative polarization to apply Corollary 7.8 below, we need to conclude that also $C\Delta$ is Cartier. This latter fact follows by boundedness and [Kol23, Thm. 5.8]; in particular, the constant C can be taken to guarantee that $C\Delta$ is Cartier and f-ample.

Then, we may invoke Corollary 7.8 for these log CY pairs and the polarization given by the Cartier divisor $C\Delta_x$. Then, $\mathbf{M}$ is pulled back from a unique space as constructed in Corollary 7.8, and the claim follows. ☐

7.2.5. *Connections to the theory of complements.* The theory of complements has been introduced by Shokurov to study flips and, more generally, morphisms of Fano type; we refer to [Sho20] for details. In recent years, this theory has proved very powerful; for instance, Birkar's proof of the BAB conjecture relies heavily on this theory; see [Bir19, Bir21]. On the other hand, in Birkar's strategy, it is important to relax the category of pairs to also include generalize pairs, and then study complements for these more general objects, originally introduced in [BZ16]. Indeed, in [Bir19], complements are built with an inductive approach, and one of the possible scenarios includes lifting complements from the base of an lc-trivial fibration with fibers of Fano type; see [Bir19, §6.4]. In particular, even when interested in pairs, the approach in [Bir19] needs to introduce generalized pairs for the inductive argument to go through. Now, by Corollary 7.7, we may induce a pair with controlled coefficients on the base of such lc-trivial fibrations. Thus, it would be interesting to explore whether boundedness of complements could be proved without resorting to generalized pairs.

7.2.6. *B-semiampleness for GLC fibrations.* Let $f\colon (Y, \Delta) \to X$ be a generically log canonical (GLC) fibration, i.e., (Y, Δ) is lc over the generic point of X. Note that, contrary to the lc-trivial case, the general fiber is no longer assumed to be log Calabi–Yau. In the GLC case, the moduli part is the b-divisor on the total space Y given by $\mathbf{M}_Y \coloneqq K_Y + \Delta - f^*(K_X + B_X)$, up to flatification of Y; see [ACSS21, §2.2] for details. It is known that $\mathbf{M}$ is b-nef, relatively b-semiample, but not b-semiample in general; see [ACSS21]. However, Shokurov conjectured that $\mathbf{M}$ is b-semiample after a small perturbation by an ample divisor coming from the moduli of the general fiber; see [Sho23, Conj. 1]. The second-named author and Spicer proved a variant of Shokurov's conjecture in [FS22, Thm. 1.2]: for GLC fibrations with klt generic fibers that are locally stable families of good minimal models, the b-divisor $\mathbf{M} + \epsilon f^* \det(f_* m\mathbf{M})$ is b-semiample, conditional to the b-semiampleness conjecture in the lc-trivial case. Here, m is sufficiently divisible and ϵ is arbitrarily small and positive. The result now holds unconditionally by Theorem 1.5.

7.3. **Moduli of Calabi–Yau varieties.** In this section we deduce Corollary 1.8. In fact, we prove a more precise statement allowing for mild singularities.

Let $\mathcal{Y}$ be a $\mathbb{G}_m$-rigidified algebraic stack of finite type parametrizing polarized klt log Calabi–Yau pairs, i.e., triples $(X, \Delta; L)$, where (X, Δ) is a klt log Calabi–Yau pair, and L is an ample line bundle on X. Note that here we rigidify with respect to the automorphisms of the line bundle L. Families of triples $(X, \Delta; L)$ are families of locally stable pairs in the sense of Kollár, together with a polarization, i.e., the datum of compatible relatively ample line bundles defined étale locally over the the base; see, e.g., [AH11, §4.2] or [Kol23, Def. 8.40].

We list noteworthy properties of the moduli stack $\mathcal{Y}$:

(i) The moduli stack $\mathcal{Y}$ is a separated (cf., [Kol23, Thm. 11.40]) Deligne–Mumford stack. Indeed, the group of polarized automorphisms of a klt log Calabi–Yau pair is finite; see, e.g., [PZ20, Prop. 10.1]. Hence, $\mathcal{Y}$ admits a coarse moduli space Y which is a separated algebraic space of finite type by [KM97].

(ii) Let $f\colon (\mathcal{X}, \Delta_{\mathcal{X}}) \to S$ be a polarized family of klt log Calabi–Yau pairs admiting a classifying map $\psi\colon S \to Y$. If f is isotrivial, then ψ is locally constant. Indeed, since the Albanese fibration of any projective klt log Calabi–Yau pair (X, Δ) is surjective and isotrivial by [PZ20, Cor. A.14], there exists an étale cover of (X, Δ), with Galois group G, which can be decomposed into a product of an abelian variety A, isogenous to the Albanese variety, and a klt log Calabi–Yau pair isomorphic to the fiber of the Albanese fibration. The G-equivariant translation group of A descends to automorphisms of (X, Δ), which identify numerically equivalent ample line bundles on X up to finite ambiguity. Therefore, for any given fixed pair (X, Δ), there are at most finitely many points of Y parametrizing triple $(X, \Delta; L)$, with an arbitrary line bundle L. This means that Y does not contain non-constant isotrivial families of pairs.

(iii) The Hodge bundle of the universal family is well-defined on $\mathcal{Y}$ as it coincides with the relative canonical bundle of the family. Its powers descend to a $\mathbb{Q}$-line bundles $M_Y^{(k)}$ on the coarse moduli space Y; see [KV04, Lem. 3.2].

(iv) The $\mathbb{Q}$-line bundles $M_Y^{(k)}$ are strictly nef by [Amb05, Thm. 3.3] or [PZ20, Thm. A.12] and (ii).

(v) There is a dense Zariski open substack $\mathcal{U} \subset \mathcal{Y}^{\mathrm{red}}$ over which the topology of the universal family of the pairs (X, Δ) (more precisely, that of the construction in Construction-Definition 6.14) is locally constant. The variation of Hodge structures associated to the universal family, as constructed in Construction-Definition 6.14, gives a period map $\phi\colon \mathcal{U} \to U \to \Gamma\backslash\mathbb{D}$, which factors through the coarse moduli space U of $\mathcal{U}$; cf. [BBT23a, Proof of Cor. 7.3].

(vi) The map $U \to \Gamma\backslash\mathbb{D}$ is quasifinite again by [Amb05, Thm. 3.3] or [PZ20, Thm. A.12] and (ii).

Corollary 7.8. *Let $\mathcal{Y}$ be an algebraic stack of finite type parametrizing polarized klt log Calabi–Yau pairs, and Y be its coarse moduli space. Then the normalization Y^ν of the reduction Y^{red} has a unique normal compactification Y^{BBH} for which some power $M_Y^{(k)}$ of the Hodge bundle of the variation of Hodge structures on middle cohomology extends to an ample bundle $\mathcal{O}_{Y^{\mathrm{BBH}}}(k)$ and such that for any family $g : Z\backslash D_Z \to \mathcal{Y}$ for a log smooth algebraic space (Z, D_Z), the resulting morphism $Z\backslash D_Z \to Y$ lifts to Y^ν and extends to extends to $\overline{g}\colon Z \to Y^{\mathrm{BBH}}$ with the property that $\overline{g}^*\mathcal{O}_{Y^{\mathrm{BBH}}}(k)$ pulls back to $(M^k_{Z\backslash D_Z})_Z$.*

Proof. The Hodge bundle of the universal family over $\mathcal{Y}$ is integrable and has torsion combinatorial monodromy by Theorem 7.1 and Theorem 7.2, since these can be checked on an alteration. These properties of the Hodge bundles, together with properties (iv) and (vi), allows us to apply Theorem 5.3: there is a normal compactification $(U^\nu)^{\mathrm{BBH}}$ of the normalization U^ν of U satisfying the properties of Theorem 5.3. In particular, the Hodge bundle $M_{U^\nu}^{(k)}$ extends amply on $(U^\nu)^{\mathrm{BBH}}$. The Hodge bundle on $\mathcal{Y}$ agrees with the Schmid extension of the Hodge bundle on a log smooth resolution $S \to \mathcal{Y}$. By the universal property of coarse moduli space, the extension $S \to (U^\nu)^{\mathrm{BBH}}$ factors through $Y^\nu \to (U^\nu)^{\mathrm{BBH}}$, which is birational and quasifinite, hence an open immersion. □

Proof of Corollary 1.8. By Bogomolov–Tian–Todorov [Bog78, Tia87, Tod89], $\mathcal{Y}$ is smooth, so Y is normal. □

7.3.1. *Stratification of Baily–Borel compactifications.* In the context of Corollary 7.8, the following description of the underlying set of points of Y^{BBH} follows from the proof in §5. For any choice of proper log smooth algebraic space (X, D) with a proper birational morphism $X\backslash D \to Y$, we will have $Y^{\mathrm{BBH}} = X(\mathbb{C})/R_{\mathrm{curve}}$.

For a particular choice (X', D'), the compactification Y^{BBH} will have a natural stratification by the images Y_S^{BBH} of the Hodge strata X'_S, and for each one there will be a period map whose projection $\widetilde{Y_S^{\mathrm{BBH}}}^{V^{\mathrm{tr}}_{S,\mathbb{Q}}} \to \mathbb{P}(V^{\mathrm{tr}}_{X',S,\mathbb{C},x_S})^{\mathrm{an}}$ has discrete fibers. Thus, for any stratum X_Σ of X whose image meets Y_S^{BBH} we have

$$\dim Y_S^{\mathrm{BBH}} \leq \operatorname{rk} \operatorname{gr}_F^{m-1} V^{\mathrm{tr}}_{X',S} \leq \operatorname{rk} \operatorname{gr}_F^{m-1} \operatorname{gr}^W_{k_\Sigma} V_{X,\Sigma} = \operatorname{rk} \operatorname{gr}_F^{m-1} \operatorname{gr}^W_{k_\Sigma} \psi_\Sigma V \tag{7.2}$$

where $\psi_\Sigma V$ is the limit mixed Hodge structure at any point of X_Σ. The last equality follows since any part of $\operatorname{gr}_F^{m-1} \operatorname{gr}^W_{k_\Sigma} \psi_\Sigma V$ is necessarily primitive, since $\operatorname{gr}_F^m \operatorname{gr}^W_{k_\Sigma+2} \psi_\Sigma V = 0$. In particular, in the case of the Baily–Borel compactification of a moduli space of d-dimensional Calabi–Yau varieties as in Corollary 7.8 (where V is the variation on degree d cohomology), we have $\dim Y = \operatorname{rk} \operatorname{gr}_F^{d-1} V$ and so

$$\operatorname{codim} Y_S^{\mathrm{BBH}} \geq \sum_{k \neq k_\Sigma} \operatorname{rk} \operatorname{gr}_F^{d-1} \operatorname{gr}^W_k \psi_\Sigma V.$$

As it can be seen from the case of $\mathcal{A}_g$ or the moduli of $K3$s, this codimension can be quite large in practice. In fact, in these two cases, equality is achieved (though in general it won't be, as either of the inequalities in (7.2) might be strict).

7.3.2. *BB vs BBH compactifications.* We give an example of a moduli stack of Calabi–Yau manifolds $\mathcal{Y}$ for which the morphism $Y^{\mathrm{BB}} \to Y^{\mathrm{BBH}}$ has positive-dimensional fibers. This is easy to do on the boundary. For example, let Y be the coarse moduli space of smooth quintic threefolds, and consider a degeneration to a transverse union of a hyperplane L and a smooth quartic threefold T, meeting along a quartic K3 surface S. The associated graded of the limit mixed Hodge structure then includes the primitive cohomology $H^3(T, \mathbb{Q})_{\mathrm{prim}}$ in weight 3 and the primitive cohomology $H^2(S, \mathbb{Q})_{\mathrm{prim}}$ in weight 2. The transcendental part of the limit mixed Hodge structure is then the transcendental part of $H^2(S, \mathbb{Q})$. Thus, since quartic threefolds and quartic surfaces satisfy an infinitesimal Torelli (for middle cohomology), and the period map of the linear systems $|\mathcal{O}_T(1)|$ on a fixed (general) T is immersive, this boundary piece survives in Y^{BB} with dimension $4 + (\binom{4+4}{4} - 5^2) = 49$. On the other hand, this boundary piece has dimension 19 in Y^{BBH}.

It is even possible for $Y^{\mathrm{BB}} \to Y^{\mathrm{BBH}}$ to contract curves in the interior of the period domain. Precisely, Y^{BB} will always contain as an open set $Y \subset \check{Y}^{\mathrm{BB}} \subset Y^{\mathrm{BB}}$ the normalization (in Y) of the closure of the image of $Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$. This is the locus of Y^{BB} where the Hodge structure does not degenerate (or equivalently, where the limit mixed Hodge structure is pure), and $\check{Y}^{\mathrm{BB}} \to Y^{\mathrm{BBH}}$ may have positive-dimensional fibers. This is indeed the case of the coarse moduli space Y of smooth quintic threefolds.

The following is an example due to Radu Laza, and we warmly thank him for sharing it with us. For a generic choice of $\underline{a} := (a_1, a_2, a_3, a_4) \in \mathbb{C}^4$,[16] the quintic threefold X in $\mathbb{P}^4$ cut by the polynomial

$$f_{\underline{a}}(x_0, x_1, x_2, x_3, x_4) := x_0^2 x_1^3 + x_0^3(x_2^2 + x_3^2 + x_4^2) + a_1 x_1^5 + a_2 x_2^5 + a_3 x_3^5 + a_4 x_4^5$$

has a unique isolated A_2 singularity at $p := [1:0:0:0:0]$. The blowup $\widetilde{X} \to X$ along p, with exceptional divisor $E \simeq \mathbb{P}(1,1,2)$, is a resolution of singularities. The Mayer–Vietoris exact sequence for the mapping cylinder of this resolution reads

$$H^{i-1}(\widetilde{X}, \mathbb{Q}) \twoheadrightarrow H^{i-1}(E, \mathbb{Q}) \to H^i(X, \mathbb{Q}) \to H^i(\widetilde{X}, \mathbb{Q}),$$

which implies that $H^i(X, \mathbb{Q})$ carries a pure Hodge structure.

Now, let $\mathcal{X}$ be the hypersurface in $\mathbb{P}^4 \times \mathbb{A}^2_{(b_1,b_2)}$ cut by

$$f_{\underline{a}}(x_0, \ldots, x_4) - (b_2^2 + b_2 + 1) b_1^2 x_1 x_0^4 + b_2(b_2+1) b_1^3 x_0^5 = 0,$$

[16] e.g., $(a_1, a_2, a_3, a_4) = (1,1,1,2)$ works but $(a_1, a_2, a_3, a_4) = (1,1,1,1)$ does not.

and let $f\colon \mathcal{X} \to \mathbb{A}^2_{(b_1,b_2)}$ be the natural projection. Note that the restriction of $\mathcal{X}$ along the curve $(b_1 = 0)$ is a trivial family with fiber X. Instead, the restriction of $\mathcal{X}$ along the curve $\{b_2 = m\}$, with m general, is a family with a unique isolated compound Du Val singularity of type cD_4 along its central fiber $\mathcal{X}_{(0,m)}$. The local model is

$$x_3^2 + x_4^2 + [x_1^3 + x_2^2 - (b_2^2 + b_2 + 1)b_1^2 x_1 + b_2(b_2 + 1)b_1^3] = 0, \tag{7.3}$$

i.e., a double suspension of [CML13, Eq. (7.3)], which is a blowup of the Weyl cover of the miniversal deformation of a cuspidal curve. By the Thom–Sebastiani's formula for Hodge modules (cf., e.g., [MSS20, Thm. 1.2]), the vanishing cohomology of (7.3) and [CML13, Eq. (7.3)] are isomorphic Hodge structures, up to a Tate shift. By [CML13, §7.7], the vanishing cohomology is then a Tate shift of the first cohomology group of an elliptic curve with j-invariant $256(b_2^2 + 3)^3/(b_2^2 - 1)^3$.

Globally, the nearby cohomology for $f\colon \mathcal{X}_{(b_1,m)} \to \mathbb{A}^1_{b_1}$ is an extension of the pure Hodge structures of the vanishing cohomology of the isolated hypersurface singularity and of the cohomology of $\mathcal{X}_{(0,m)}$, as one can check taking cohomology of the specialization triangle [PS08, p.276] and using the purity of the vanishing cohomology and of $H^*(X, \mathbb{Q})$. By the purity of the limiting mixed Hodge structure, the period map $Y^{\mathrm{an}} \to \Gamma\backslash\mathbb{D}$ extends to $(\mathbb{A}^2)^{\mathrm{an}}_{(b_1,b_2)} \to \Gamma\backslash\mathbb{D}$.

We conclude that the curve $(b_1 = 0)$ maps generically finitely in $\breve{Y}^{\mathrm{BB}}$, but it is contracted by $Y^{\mathrm{BB}} \to Y^{\mathrm{BBH}}$. Indeed, the Hodge bundle is trivial along $(b_1 = 0)$ since f is trivial along the curve, but the Griffiths bundle of $R^3 f_* \mathbb{Q}_{\mathcal{X}}|_{(\mathbb{A}^1)^*_{b_1} \times \mathbb{A}^1_{b_2}}$ is positive since the j-invariant of the vanishing cohomology as a summand of the nearby cohomology varies.

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