p-adic hyperbolicity for Shimura varieties and period images

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Abstract

We prove that Shimura varieties and geometric period images satisfy a p-adic extension property for large enough primes p. More precisely, let $\mathsf{D}^\times \subset \mathsf{D}$ denote the inclusion of the closed punctured unit disc in the closed unit disc. Let X be either a Shimura variety or a geometric period image with torsion-free level structure. Let F be a discretely valued p-adic field containing the number field of definition of X, where p is a large enough prime. Then, any rigid-analytic map $f : (\mathsf{D}^\times)^a \times \mathsf{D}^b \to X_F^{\mathrm{an}}$ defined over F whose image intersects the good reduction locus of X_F^{an} (with respect to an integral canonical model) extends to a map $\mathsf{D}^{a+b} \to X_F^{\mathrm{an}}$. We note that this hypothesis is vacuous if X is proper. We also deduce an application to algebraicity of rigid-analytic maps. Our methods also apply to the more general situation of the rigid generic fiber of formal schemes admitting Fontaine-Laffaile modules which satisfy certain positivity conditions.

1 Introduction

The purpose of this paper is to prove p-adic extension and algebraicity theorems for exceptional Shimura varieties and geometric period images. This result is a p-adic analogue of the following theorems for complex Shimura varieties that Borel ([Bor72]) proved in 1972:

Theorem (Borel extension)**.** *Let $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$ be a Shimura variety with torsion-free level structure. Let D be the complex open disc and let $D^\times$ be the punctured open unit disc. Then, every holomorphic map ${D^\times}^a \times D^b \to \mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})^{\mathrm{hol}}$ extends to a map $D^{a+b} \to (\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})^{\mathrm{BB}})^{\mathrm{hol}}$.*

An immediate corollary of this extension result and GAGA is the following algebraicity theorem.

Theorem (Borel algebraicity)**.** *Let $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$ be as above, and let M be a complex algebraic variety. Then every holomorphic map $M^{\mathrm{hol}} \to \mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})^{\mathrm{hol}}$ is the analytification of an algebraic map $M \to \mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$.*

Here is the main theorem of this paper.

Theorem 1.1. *Let X be either a Shimura variety or a geometric period image with torsion-free level structure. There exists an integer N with the following property. Let p be a prime that doesn't divide N and suppose F is a discretely valued p-adic field containing the field of definition of X. Suppose that $f : (\mathsf{D}^\times)^a \times \mathsf{D}^b \to X_F^{\mathrm{an}}$ is a rigid-analytic map defined over F such that $\mathrm{Im}(f)$ intersects the good-reduction locus of X_F. Then, f extends to a map $\mathsf{D}^{a+b} \to X_F^{\mathrm{an}}$.*

Theorem 1.1 has the following corollary.

Theorem 1.2. *Let X, p and F be as above, and let M be an algebraic variety defined over F. Then, every rigid-analytic map $f : M^{\mathrm{an}} \to X^{\mathrm{an}}$ defined over F such that $\mathrm{Im}(f)$ is contained in the good reduction locus is the analytification of an algebraic map $M \to X$.*

Remark 1.3.

1. We define the good reduction locus precisely in Section 3, but informally, the good reduction locus is the analytic open subspace of X^{an} whose classical K-points arise as $\mathcal{O}_K$-points of a good integral model of X. If X is proper, then the good reduction locus is all of X^{an} and therefore the good reduction hypothesis is vacuous. We expect the theorem to hold without this hypothesis.

2. We draw the reader's attention to the fact that the good reduction hypothesis in Theorem 1.1 has the consequence that the extension of f yields a map to X, and it is not necessary to compactify X.

3. The setting of Shimura varieties of *abelian type* is addressed in [OSZP24], where the authors prove p-adic extension and algebraicity theorems without a good reduction hypothesis and for all primes p. However, the extension of the map from $(\mathsf{D}^\times)^a \times \mathsf{D}^b$ is only obtained to the Baily-Borel compactification of X. Indeed, one may start with a map from D to the Baily-Borel compactification with the property that $\mathsf{D}^\times$ maps to the interior and 0 maps to the boundary.

1.1 Other results

The proofs of Theorem 1.1 and 1.2 work in a more general setting than just the case of geometric period images and Shimura varieties. In order to not mire ourselves in unenlightening notation and technicalities, we will state a result that is not the most general but that is the cleanest to state.

Theorem 1.4. *Let $\mathscr{X}$ be a smooth scheme over $W(\overline{\mathbb{F}}_p)$, and let $\mathscr{X}^{\mathrm{rig}}$ denote its rigid generic fiber. Let $\mathbb{L}/\mathscr{X}^{\mathrm{rig}}$ be a crystalline local system with $\mathbb{V}_{\mathrm{FL}}$ the associated Fontaine-Laffaile module. Suppose that we are in one of the following two cases.*

1. *The Kodaira-Spencer map associated to the filtered flat bundle underlying $\mathbb{V}_{\mathrm{FL}}$ is everywhere immersive.*

2. *The Griffiths bundle associated to $\mathbb{V}_{\mathrm{FL}}$ is an ample bundle on $\mathscr{X}$.*

Then, every map $(\mathsf{D}^\times)^a \times \mathsf{D}^b \to \mathscr{X}^{\mathrm{rig}}$ extends to a map $\mathsf{D}^{a+b} \to \mathscr{X}^{\mathrm{rig}}$

1.2 Outline of proof

The main results of [OSZP24] proved the p-adic extension and algebraization results for Shimura varieties *of abelian type* parallelling Borel's theorem in the setting of a discretely valued p-adic field. The strategy of [OSZP24] crucially uses the existence of Rapoport-Zink ([RZ96]) spaces and Rapoport-Zink uniformizations of $\mathcal{A}_g$. This in turn of course relies on the moduli interpretation of $\mathcal{A}_g$. While there is a theory of Rapoport-Zink spaces (see [RV14]) that goes beyond the setting of abelian varieties, it is not known (though it is certainly expected) that exceptional Shimura varieties admit such uniformization maps. The setting of geometric period images is even more barren, without even any expectations of p-adic uniformization maps. Our proof therefore sidesteps the existence Rapoport-Zink uniformizations and instead make strong use of the existence of crystalline local systems and Fontaine-Laffaile modules.

The outline of our proof is as follows. The main step is the case of a one-dimensional disk. For brevity, we will focus on the Shimura case. We work at a prime p at which X has an integral canonical model (which we will denote by $\mathscr{X}$ in the introduction)—$\mathscr{X}$ is equipped with ℓ-adic local systems and a Fontaine-Laffaille module, and X is equipped with a crystalline p-adic local system associated to the Fontaine-Laffaille module. We first prove that any map $f : \mathsf{D}^\times \to X_F^{\mathrm{an}}$ has the property that $f(\mathsf{D}^\times)$ is either entirely contained in the good reduction locus, or the bad reduction locus. This proof is ℓ-adic and follows the arguments in [OSZP24], and uses a monodromy-theoretic description of the good-reduction locus proved in [PST+21] for exceptional Shimura varieties. By a recent result in [DY25], we know that the p-adic local system extends to D. We then apply the theory of prismatic F-crystals to show that up to shrinking D, the F-crystal associated to the crystalline Galois representation $\mathbb{L}_x$ is independent of the classical point $x \in \mathsf{D}$. Now, we write $\mathsf{D}^\times$ as an increasing union of annuli A_k, each of which admits an integral model $\mathfrak{A}_k$ that maps to $\mathscr{X}$. We then generalize an argument of Oort ([Oor04]) to show that the F-crystal on $\mathscr{X}$ mod p pulls back to $\mathfrak{A}_k$ mod p. Finally, we use the Kodaira-Spencer map to prove that any map from a connected variety over $\overline{\mathbb{F}}_p$ to $\mathscr{X}$ mod p with the property that the F-crystal over $\mathscr{X}$ mod p pulls back to something constant must in fact be the constant map. We then conclude that the map from $\mathsf{D}^\times$ to X_F^{an} maps to a residue disc, and therefore extends by the Riemann extension theorem.

To deduce Theorem 1.1 for morphisms from polydisks $f : (\mathsf{D}^\times)^a \times \mathsf{D}^b \to X_F^{\mathrm{an}}$, we show that the existence of an extension on any one-dimensional disk implies that f extends meromorphically and then use the p-adic Riemann–Hilbert correspondence of [DLLZ23] to show that the exceptional fibers in the resolution of indeterminacies of f must be contracted. In fact, this part of the argument also shows that the one-dimensional disk case of Theorem 1.1 without the good reduction assumption implies the polydisk case (without the good reduction assumption).

1.3 Previous work

There are several results prior (aside from Borel's work) to our work that addresses the questions of algebraicity and extension – both in the complex and p-adic settings. In the complex case, [BBT23] and [BFMT25] prove the algebraicity and extension results for geometric period images.

As earlier mentioned, [OSZP24] treats the case of abelian Shimura varieties for all primes p, without a good-reduction hypothesis. It also treats the case of the universal abelian scheme over compact Shimura varieties of Hodge type, and Rapoport-Zink spaces associated to $\mathcal{A}_{g,\mathsf{K}}$. The paper [OP25] proves the p-adic extension theorem for local Shimura varieties. Cherry (in [Che02]) addresses the case of genus ≥ 2 curves in the more general situation of $\mathbb{C}_p$. Cherry-Ru ([CR04]) prove a p-adic big Picard style theorem, and Sun ([Sun20]) proves the $\mathbb{C}_p$-analogue of the algebraicity theorem.

1.4 Organization of the paper

In Section 2, we introduce various objects that live on Shimura varieties and period images. In Section 3, we prove that the image of every map $\mathsf{D}^\times \to X$ must either be entirely contained in the good reduction locus or the bad reduction locus. In Section 4, we recall results about prismatic F-crystals, and in Section 5 prove a crucial constancy result for F-crystals. In Section 6, we generalize work of Oort to show that a pointwise constant F-crystal on $\mathbb{P}^1$ must be constant. In Section 7, we prove the main theorem for $\mathsf{D}^\times$, and prove the main theorem in general for 8.

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2 Notations and the general setup

2.1 The set-up for the Shimura case

We follow closely the notations of [BST24]. To recall, $(G, \mathbf{X})$ shall denote a Shimura datum. Let $E := E(G, \mathbf{X}) \subset \mathbb{C}$ denote the reflex field. We let V be the adjoint representation of G, a $\mathbb{Z}$-lattice $\mathbb{V}$ of V, and a neat compact open subgroup $\mathsf{K} \subset G(\mathbb{A}_f)$ that stabilizes $\mathbb{V} \otimes \hat{\mathbb{Z}} \subset V \otimes_{\mathbb{Q}} \mathbb{A}_f$. We further assume that K acts trivially on $\mathbb{V}/3\mathbb{V}$. Denote by $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$, the corresponding Shimura variety over E. Associated to $\mathbb{V} \subset V$, we have a family of $\mathbb{Z}_\ell$ (respectively $\mathbb{Q}_\ell$) étale local systems on $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$ that we denote by $\mathbb{V}_{\text{ét},\ell}$ (resp. $V_{\text{ét},\ell}$.) We denote by $V_{\mathrm{dR}} := (\mathcal{V}, \nabla, F^\bullet\mathcal{V})$ the associated filtered flat vector bundle on $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$ defined over E.

We pick a large integer N as in [BST24, Theorem 1.3], so that $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$ admits a smooth model $\mathscr{S}_{\mathsf{K}}(G, \mathbf{X})$ over $\mathcal{O}_E[1/N]$ and such that for all places v of E outside N, $\mathscr{S}_{\mathsf{K}}(G, \mathbf{X}) \otimes_{\mathcal{O}_E[1/N]} \mathcal{O}_{E_v}$ is a *canonical* integral model of $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X}) \otimes_E E_v$ over the ring of integers $\mathcal{O}_{E_v}$ of the v-adic completion E_v of E. Furthermore, for every place $v \nmid N$, $\mathscr{S}_{\mathsf{K}}(G, \mathbf{X}) \otimes_{\mathcal{O}_E[1/N]} \mathcal{O}_{E_v}$ admits a log-smooth compactification over $\mathcal{O}_{E_v}$. The $\mathbb{Z}_\ell$-étale local systems $\mathbb{V}_{\text{ét},\ell}$ on $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X}) \otimes_E E_v$, extend to ℓ-adic étale local systems on $\mathscr{S}_{\mathsf{K}}(G, \mathbf{X}) \otimes_{\mathcal{O}_E[1/N]} \mathcal{O}_{E_v}[1/\ell]$. We also have that for $p \nmid N$, the restriction of $\mathbb{V}_{\text{ét},p}$ to $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X}) \otimes_E E_v$ is crystalline in the sense of Faltings-Fontaine-Laffaille, where v is a place of E dividing p. By increasing N if necessary, we may also assume that $(\mathcal{V}, \nabla)_{\mathrm{dR}}$ spreads out to $\mathscr{S}_{\mathsf{K}}(G, \mathbf{X})$ such that the Kodaira-Spencer map is everywhere immersive.

We shall fix henceforth a rational prime $p \nmid N$, a place v of E above p. We shall work throughout over the p-adic local field $K := E_v$. Set $\mathscr{S} := \mathscr{S}_{\mathsf{K}}(G, \mathbf{X}) \otimes_{\mathcal{O}_E[1/N]} \mathcal{O}_{E_v}$, and $S := \mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X}) \otimes_E E_v$. By a slight abuse of notation, we denote the pullbacks to S of the local systems $V_{\text{ét},\ell}$, $\mathbb{V}_{\text{ét},\ell}$ and the filtered flat vector bundle V_{dR} also by $V_{\text{ét},\ell}$, $\mathbb{V}_{\text{ét},\ell}$ and V_{dR} respectively, and in the case $\ell \neq p$, their extensions to the integral canonical model $\mathscr{S}$ shall also be denoted by the same. We let $\mathbb{V}_{\mathrm{FL}}$ denote the Fontaine-Laffaile module associated to $\mathbb{V}_{\text{ét},p}/S$ on the formal p-adic completion $\hat{\mathscr{S}}$, and we let $\mathbb{V}_{\mathrm{cris}}$ denote the F-crystal on $\mathscr{S}_p := \mathscr{S} \otimes_{\mathcal{O}_{E_v}} k_v$, where k_v is the residue field of $\mathcal{O}_{E_v}$. Note that the filtered flat bundle underlying $\mathbb{V}_{\mathrm{FL}}$ is just $(\mathcal{V}, \nabla)_{\mathrm{dR}}$. We will sometimes use the symbol $\mathbb{L}$ to denote the $\mathbb{Z}_p$-étale local system $\mathbb{V}_{\text{ét},p}$ on S We denote by $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})^{\mathrm{BB}}$ the Baily–Borel compactification of $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$ and set $S^{\mathrm{BB}} := \mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})^{\mathrm{BB}} \otimes_E E_v$.

2.2 The set-up for geometric period images

Let $E \subset \mathbb{C}$ be a number field, P a smooth, connected quasi-projective algebraic variety over E, and $f : Z \to P$ a smooth, projective E-morphism. For a fixed m, we have a polarizable integral variation of Hodge structures $(\mathbb{W}_{\mathbb{Z}}, F^\bullet)$ with underlying $\mathbb{Z}$-local system $\mathbb{W}_{\mathbb{Z}} := R^m f_*^{\mathrm{hol}}(\underline{\mathbb{Z}}_{Z^{\mathrm{hol}}})$. Denote by G, the generic Mumford-Tate group of the variation. We shall further assume that the variation has *neat monodromy*, which can always be arranged after passing to a finite étale cover of P. We denote by $\mathbb{W}_{\text{ét},\ell} := R^m f_*^{\text{ét}}(\underline{\mathbb{Z}_\ell})$ the associated $\mathbb{Z}_\ell$ local system on P, and by $W_{\text{dR}} := (\mathcal{W}, \nabla, F^\bullet)$ the filtered flat algebraic vector bundle on P defined over E, such that $\mathbb{W}_{\mathbb{Z}}$ is the sheaf of flat sections of the associated analytified filtered flat bundle on P^{hol}.

We denote by $Y_{/E}$ the Stein factorization of the period map associated to the variation $(\mathbb{W}_{\mathbb{Z}}, F^\bullet)$, in the sense of [BST24, §1.4] (see also [BBT23]). Thus, $Y_{/E}$ is a quasi-projective algebraic variety over E, and the associated period map $\phi : P_{\mathbb{C}}^{\text{hol}} \to \Gamma\backslash D$, factors as a composite $P^{\text{hol}} \xrightarrow{g^{\text{hol}}} Y_{\mathbb{C}}^{\text{hol}} \to \Gamma\backslash D$, where the morphism $Y_{\mathbb{C}}^{\text{hol}} \to \Gamma\backslash D$ is finite, and $P \xrightarrow{g} Y_{/E}$ is an algebraic map defined over E with geometrically connected generic fiber. There also exists a smooth partial compactification P' of P defined over E and a proper map $P' \to Y$, with W_{dR}, $\mathbb{W}_{\text{ét},\ell}$ (resp. $(\mathbb{W}_{\mathbb{Z}}, F^\bullet)$) extending to P' (resp. $P'^{,\text{hol}}$).

Note that the variation $(\mathbb{W}_{\mathbb{Z}}, F^\bullet)$ descends to a polarizable $\mathbb{Z}$ variation of Hodge structures $(\mathbb{V}_{\mathbb{Z}}, F^\bullet)$ on $Y_{\mathbb{C}}^{\text{hol}}$, as do the $\mathbb{Z}_\ell$-étale local systems $\mathbb{W}_{\text{ét},\ell}$ to $\mathbb{Z}_\ell$-étale local systems $\mathbb{V}_{\text{ét},\ell}$ on $Y_{/E}$ (see [BST24, §2.6]). The filtered flat bundle $W_{\text{dR}} = (\mathcal{W}, \nabla, F^\bullet)$ on P descends to a filtered vector bundle on $Y_{/E}$ defined over E, denoted by $V_{\text{dR}} = (\mathcal{V}, F^\bullet)$.

There is a finite set of places Σ of E such that:

- $Z \to P$ spreads out to a smooth proper family over a smooth base $\mathcal{Z} \to \mathcal{P}$ over $\mathcal{O}_{E,\Sigma}$.
- $P' \to Y$ spreads out to a proper map $\mathcal{P}' \to \mathcal{Y}$ over $\mathcal{O}_{E,\Sigma}$ where $\mathcal{P}'$ is a smooth partial compactification of $\mathcal{P}$.
- The filtered flat bundle W_{dR}/P' and the filtered bundle V_{dR}/Y spread out to a filtered flat bundle on $\mathcal{P}'$ and a filtered bundle on $\mathcal{Y}$. We abusively denote the extensions by the same notation. The Griffiths bundle of V_{dR} on $\mathcal{Y}$ is ample.
- For every prime ℓ, the local system $\mathbb{W}_{\text{ét},\ell}$ extends to an ℓ-adic local system on $\mathcal{P}'_{\mathcal{O}_{E,\Sigma_\ell}}$ where Σ_ℓ is the union of Σ and all primes of E dividing ℓ. We abusively denote these local system by $\mathbb{W}_{\text{ét},\ell}$ as well. Likewise, the $\mathbb{Z}_\ell$-étale local systems $\mathbb{V}_{\text{ét},\ell}$ on Y extend to $\mathbb{Z}_\ell$-local systems on $\mathcal{Y}_{\mathcal{O}_{E,\Sigma_\ell}}$, which we denote by the same notation.
- $\mathcal{Y}$ is an integral canonical model (as in [BST24]) of Y over $\mathcal{O}_{E,\Sigma}$.
- There is a uniform stratified resolution with boundary $\mathcal{S}^j \to \mathcal{Y}^j$ of $\mathcal{Y}$ over $\mathcal{O}_{E,\Sigma}$ as in [BST24, Definition 4.2], and for each j, there is a smooth scheme $\mathcal{T}^j/\mathcal{O}_{E,\Sigma}$ and maps $\mathcal{T}^j \xrightarrow{t^j} \mathcal{P}'$ and $\mathcal{T}^j \xrightarrow{q^j} \mathcal{S}^j$ as in [BST24, Section 5.2]. For the largest-dimesnional stratum $\mathcal{Y}^m = \mathcal{Y}$, we may assume $\mathcal{P}' \to \mathcal{Y}$ factors through $\mathcal{S}^m$, and that $\mathcal{P}' \to \mathcal{S}^m$ has geometrically connected fibers.
- We let $\mathbb{U}^j_{\text{ét},\ell}$ denote the pullback of $\mathbb{V}_{\text{ét},\ell}|_{\mathcal{Y}^j \times \mathcal{O}_{E,\Sigma_\ell}}$ to $\mathcal{S}^j \times \mathcal{O}_{E,\Sigma_\ell}$, and let $(\mathcal{U}, \nabla)_{\text{dR}}{}^j$ be the pullback of $V_{\text{dR}}|_{\mathcal{Y}^j}$—note that $(\mathcal{U}, \nabla)_{\text{dR}}{}^j$ is a filtered flat bundle.

The filtered flat bundle $W_{\text{dR}}/\mathcal{P}'$ has the structure of a Fontaine-Laffaille module at a prime $v \notin \Sigma$ and this corresponds to the local system $\mathbb{W}_{\text{et},p}$ for $v \mid p$ via the Faltings-Fontaine-Laffaile correspondence. There are two filtered flat bundles on $\mathcal{S}^j$—one is $(\mathcal{U}, \nabla)_{\text{dR}}{}^j$, already defined. The other one is the filtered flat bundle underlying the Fontaine-Laffaille module associated to $\mathbb{U}^j_{\text{ét},p}$. We denote this Fontaine-Laffaille module by $\mathbb{U}^j_{\text{FL}}$. We note that both these filtered flat bundles become isomorphic when pulled back to $\mathcal{T}^j$. Note however that for the largest stratum $\mathcal{S}^m$, these two filtered flat vector bundles *are* isomorphic, since $\mathcal{P}' \to \mathcal{S}^m$ has geometrically connected fibers.

Henceforth, we fix once and for all a finite place $v \notin \Sigma$. Let p denote the rational prime below v. Set $K := E_v$, with ring of integers $\mathcal{O}_K$ and residue field k. We will let Y, P, V_{dR} (resp. $\mathcal{Y}, \mathcal{S}, \mathcal{T},$) etc. also denote the basechange of the corresponding objects from E (resp. $\mathcal{O}_{E,\Sigma}$) to K (resp. $\mathcal{O}_K$). Note that the p-adic (and ℓ-adic) local systems on all the spaces agree under pullback. We will sometimes use the symbol $\mathbb{L}$ to denote our p-adic local system(s) if the base is implicit (or unimportant). We denote by Y^{BB} the Baily–Borel compactification of the period image Y (see [BFMT25]).

2.3 General notations

As above, we fix a rational prime p, a p-adic field K with ring of integers $\mathcal{O}_K$, and residue field k.

For a complete non-archimedean field extension F of K, rigid-analytic varieties and spaces over F shall be viewed as adic spaces over $\mathrm{Spa}(F, \mathcal{O}_F)$. In particular, by a rigid-analytic variety over F, we shall mean a quasi-separated adic space that is locally of finite type over $\mathrm{Spa}(F, \mathcal{O}_F)$. For an admissible formal $\mathcal{O}_F$-scheme $\mathscr{X} / \mathrm{Spf}(\mathcal{O}_F)$, we denote the associated rigid-analytic generic fiber by $\mathscr{X}^{\mathrm{rig}}$. For an algebraic variety X over F (respectively a morphism $g : W \to X$ of algebraic varieties over F), we denote by X^{an} the associated rigid-analytic space over $\mathrm{Spa}(F)$ (resp. by $g^{\mathrm{an}} : W^{\mathrm{an}} \to X^{\mathrm{an}}$ the associated morphism of rigid-analytic spaces over F).

For a complex algebraic variety $X \to \mathrm{Spec}(\mathbb{C})$ (respectively a morphism of complex algebraic varieties $g : W \to X$) we denote the associated complex analytic space by X^{hol} (respectively the associated morphism of complex analytic spaces by g^{hol}).

The rigid-analytic closed unit disk over F is denoted by $\mathsf{D}_F := \mathrm{Spa}(F\langle t\rangle, \mathcal{O}_F\langle t\rangle)$. The punctured closed unit disk over F is $\mathsf{D}_F^{\times} := \mathsf{D}_F \setminus \{t = 0\}$.

3 Boundary and interior

Definition 3.1. Let X denote either the Shimura variety S or the period image Y, and let $\mathscr{X}$ denote the integral canonical model. Let K be a discretely valued field with ring of integers $\mathcal{O}_K$. We say that a point $x \in \mathscr{X}(K)$ has good reduction if its specialization lies in the interior, i.e. x is induced by an $\mathcal{O}_K$-valued point of $\mathscr{X}$. We say that x has bad reduction otherwise. Define the *good reduction locus* X^{good} of X to be the set of points of X^{an} whose mod p specialization with respect to the canonical model lies in the interior. We define the bad reduction locus to be the complement in X^{an} of the good reduction locus. We note that X^{good} is an analytic open subspace of X^{an}.

We will first prove that a map $f : \mathsf{D}^{\times} \to X^{\mathrm{an}}$ must either be contained entirely in the good reduction locus or the bad reduction locus where X is as above. This argument appears in an old arXiv version of [BST24] (Lemma 6.4) (which is turn is essentially the same as the argument in the abelian case [OSZP24, Theorem 3.3]) but we include it in this paper for completion.

Theorem 3.2. *Let $f : \mathsf{D}^{\times} \to X^{\mathrm{an}}$ be an analytic map where X is as above. Then either $f(\mathsf{D}^{\times}) \subseteq X^{\mathrm{good}}$ or $f(\mathsf{D}^{\times}) \subseteq (X^{\mathrm{an}} \setminus X^{\mathrm{good}})$.*

The following lemma is an analogue of the Neron-Ogg-Shafarevich criterion and follows directly by the arguments of [PST$^+$21, Lemma 8.4].

Lemma 3.3. *Let K either be $\mathbb{F}((t))$ or a discretely valued p-adic field, and let $x \in X(K)$ be a point. Then x has bad reduction if and only if the action of the inertia subgroup $I_K \subset \mathrm{Gal}_K$ on $(\mathbb{V}_{\acute{e}t,\ell})_x$ is quasi-unipotent of infinite order.*

Proof of Theorem 3.2. [OSZP24, Proposition 3.6] does what is required for thin annuli. To deduce the result for thick annuli from thin annuli, we proceed along identical lines to the argument in [OSZP24, Section 3.1.2]. We are reduced to proving the following result. Let $g : \mathbb{G}_m \to \mathscr{X}_p$ be any map. Then, g extends to a map $\mathbb{P}^1 \to \mathscr{X}_p$. To prove this, it suffices to prove that the local monodromy of $g^{-1}\mathbb{V}_{\text{ét},\ell}$ around the boundary points is semi-simple. To prove this, it suffices to prove that the geometric monodromy of $g^{-1}\mathbb{V}_{\text{ét},\ell}$ is semisimple. By [Del80, 3.4.12], it suffices to prove that the arithmetic local system $g^{-1}\mathbb{V}_{\text{ét},\ell}$ is point-wise pure—of course, this would follow from proving that $\mathbb{V}_{\text{ét},\ell}$ itself were pure.

In the Shimura case, this follows from the fact that $\mathbb{V}_{\text{ét},\ell}$ is induced by the adjoint representation of G, and is therefore an irreducible local system with finite determinant. In the case of geometric period maps, this follows from the fact that the local system is induced by a geometric family, and is therefore point-wise pure by [Del80].

□

4 Crystalline p-adic local systems and analytic prismatic F-crystals

In this section, we recall the notions of crystalline local systems and F-crystals that are used in the article.

Definition 4.1. Let X/k be a smooth scheme. Let X_{crys} denote the p-completed crystalline site of X,[1] equipped with the structure sheaf $\mathcal{O}_{X,\text{crys}}$.

(i) By a *crystal* over X we mean a *finite locally free crystal* or equivalently, a *crystal of vector bundles* over X, that is, a sheaf of $\mathcal{O}_{X,\text{crys}}$-modules $\mathbb{E}$ such that for each PD-thickening (U, T) in X_{crys}, the induced Zariski sheaf $\mathbb{E}_T$ is a finite locally free $\mathcal{O}_T$-module, such that for each morphism $g : (U', T') \to (U, T)$ in X_{crys}, the induced map $g^*\mathbb{E}_T \xrightarrow{\sim} \mathbb{E}_{T'}$ is an isomorphism.

(ii) An *isocrystal* over X is an object in the isogeny category of crystals of modules. All of the isocrystals we will consider will in fact be obtained from a crystal (in vector bundles) by inverting p.

(iii) An *F-crystal* (resp. *F-isocrystal*) over X consists of a pair $(\mathbb{E}, \varphi)$ where $\mathbb{E}$ is a crystal (resp. isocrystal) over X and φ is an isomorphism

$$\varphi : F_{\text{crys}}^*\mathbb{E}[1/p] \xrightarrow{\sim} \mathbb{E}[1/p]$$

which is compatible with the Frobenius map F_{crys} on $\mathcal{O}_{X,\text{crys}}$ induced by functoriality.

We write $\text{Vect}^\varphi(X_{\text{crys}})$ (resp. $\text{Isoc}^\varphi(X_{\text{crys}})$) for the category of F-crystals (resp. F-isocrystals) over X.

We also need the notion of prismatic and analytic prismatic F-crystals. Let us first recall that, given a p-adic formal scheme $\mathfrak{X}/\mathcal{O}_K$, its absolute prismatic site $\mathfrak{X}_{\Delta}$ is the opposite of the category of bounded prisms (A, I) equipped with a map $\operatorname{Spf} A/I \to \mathfrak{X}$, endowed with the flat topology (on prisms). Let $\mathcal{O}_{\Delta}$ (resp. $\mathcal{I}_{\Delta}$) denote the structure sheaf (resp. the Hodge–Tate sheaf) on $\mathfrak{X}_{\Delta}$, which sends $(A, I) \mapsto A$ (resp. sends $(A, I) \mapsto I$). Let φ_{Δ} denote the Frobenius map on $\mathcal{O}_{\Delta}$.

Example 4.2. Let $E = E(u)$ be an Eisenstein polynomial for a fixed uniformizer $\varpi \in \mathcal{O}_K$.

[1] In this article, we shall consider the *absolute* crystalline site of X, or equivalently, crystalline site of X over the divided power algebra $(W(k), p)$.

1. The *Breuil–Kisin* prism $(\mathfrak{S}, E)$ with $\mathfrak{S} = W(k)[\![u]\!]$ and $\varphi_{\mathfrak{S}}(u) = u^p$ gives an object in $(\operatorname{Spf} \mathcal{O}_K)_{\Delta}$ via the surjection $\mathfrak{S} \to \mathcal{O}_K$ sending $u \mapsto \varpi$. In fact, by the argument in [BS23, Example 2.6(1)] it covers the final object of the topos $\operatorname{Shv}((\operatorname{Spf} \mathcal{O}_K)_{\Delta}, \mathcal{O}_{\Delta})$. Taking the Cech nerve of $(\mathfrak{S}, E)$ over the final object in this topos gives rise to a cosimplicial object
$$\mathfrak{S} \rightrightarrows \mathfrak{S}^{(1)} \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} \mathfrak{S}^{(2)} \tag{1}$$
in $(\operatorname{Spf} \mathcal{O}_K)_{\Delta}$. One can explicitly describe the Prisms $(\mathfrak{S}^{(i)}, E)$ in terms of the prismatic envelop construction. For example, we have $\mathfrak{S}^{(1)} = W[\![u, v]\!]\left\{\frac{u-v}{E}\right\}^{\wedge}_{(p,E)}$, where $\{\cdot\}$ denotes the prismatic envelop.

2. Let $R = (\mathcal{O}_K[t])^{\wedge}_p$ be the p-adic completion of $\mathcal{O}_K[t]$ and let $R_0 = (W[t_0])^{\wedge}_p$. Let $\mathfrak{X} = \operatorname{Spf} R$ be the p-adic formal $\mathbb{A}^1$ over $\operatorname{Spf} \mathcal{O}_K$. Let $\mathfrak{S}_R = R_0[\![u]\!]$, equipped with a δ-structure given by $\varphi(u) = u^p$ and $\varphi(t_0) = t_0^p$. As in the previous example, we have a surjection $\mathfrak{S}_R \to R$ sending $u \mapsto \varpi$ and $t_0 \mapsto t$, which makes $(\mathfrak{S}_R, E)$ into an object in $\mathfrak{X}_{\Delta}$.

Lemma 4.3. *In the second example above, the prism $(\mathfrak{S}_R, E)$ covers the final object in the topos of $(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$.*

Proof. Let (A, I) be a prism in the absolute prismatic site $\mathfrak{X}_{\Delta}$, equipped with a map $\iota : R \to A/I$. Note that A is canonically a W-algebra. Let us pick an element $\widetilde{\varpi}$ (resp. $\widetilde{t}$) in A which lifts the image of ϖ (resp. of t) in A/I under ι. Let us consider the map $A \to (A \otimes_W \mathfrak{S}_R)^{\wedge}_{(p,I)}$ of δ-rings and form the prismatic envelope (see [BS22, Proposition 3.13])
$$B := (A \otimes_W \mathfrak{S}_R)\{\frac{u - \widetilde{\varpi}, t_0 - \widetilde{t}}{I}\}^{\wedge}_{(p,I)},$$
so we have a map $(A, I) \to (B, IB)$ of prisms in $\mathfrak{X}_{\Delta}$. By [BS22, Proposition 3.13], the map $A \to B$ is (p, I)-completely flat, and in fact (p, I)-completely faithfully flat, so $(A, I) \to (B, IB)$ forms a cover in $\mathfrak{X}_{\Delta}$. Finally, note that by construction we have $E(u) = E(\widetilde{\varpi}) = 0 \mod IB$, so we have $E(u) \in IB$ and thus we have a map of prisms $(\mathfrak{S}_R, E(u)) \to (B, IB)$ by [BS22, Lemma 2.24]. This finishes the proof of the lemma. □

Definition 4.4. Let $\mathfrak{X}/\mathcal{O}_K$ be a smooth p-adic formal scheme. A *prismatic crystal (of vector bundles)* over $\mathfrak{X}$ is an $\mathcal{O}_{\Delta}$-module $\mathcal{E}$, such that there exists bounded prisms (A_i, I_i) in $\mathfrak{X}_{\Delta}$, with $\{U_i = \operatorname{Spf}(A_i/I_i)\}$ covering the final object of the topos of $(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$, and finite projective A_i-modules E_i, such that $\mathcal{E}|_{U_i} \cong E_i \otimes_{A_i} \mathcal{O}_{\Delta, U_i}$. A *prismatic F-crystal* over $\mathfrak{X}$ is pair $(\mathcal{E}, \varphi_{\mathcal{E}})$, where $\mathcal{E}$ is a prismatic crystal and $\varphi_{\mathcal{E}}$ is an isomorphism
$$\varphi_{\mathcal{E}} : \varphi_{\Delta}^* \mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$$
of $\mathcal{O}_{\Delta}$-modules. We write $\operatorname{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$ (resp. $\operatorname{Vect}^{\varphi}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$) for the category of prismatic crystals (resp. prismatic F-crystals) over $\mathfrak{X}$.

Following notations from [BS23], for a given bounded prism (A, I), we write $\operatorname{Vect}(A)$ for the category of finite projective A-modules. We write $\operatorname{Vect}^{\varphi}(A, I)$ for the category of pairs (E, φ_E) that consists of a finite projective A-module E together with an A-linear isomorphism $\varphi_E : \varphi_A^* E[1/I] \xrightarrow{\sim} E[1/I]$, with morphisms being morphisms between the finite projective A-modules that are compatible with Frobenius. By (p, I)-completely faithfully flat descent for vector bundles (see [BS23, Proposition 2.7]), we have natural equivalences

$$\mathrm{Vect}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \lim_{(A,I)\in\mathfrak{X}_{\Delta}} \mathrm{Vect}(A)$$
$$\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \lim_{(A,I)\in\mathfrak{X}_{\Delta}} \mathrm{Vect}^{\varphi}(A, I)$$

In particular, to specify a prismatic F-crystal over $\mathfrak{X}$ is equivalent to specifying a prismatic F-crystal over each bounded prism $(A, I) \in \mathfrak{X}_{\Delta}$ in a compatible fashion. If $(A_0, I_0) \in \mathfrak{X}_{\Delta}$ is a bounded prism that covers the final object of $\mathrm{Shv}(\mathfrak{X}_{\Delta}, \mathcal{O}_{\Delta})$, this is also equivalent to the data of a prismatic F-crystal over (A_0, I_0) which satisfies certain the descent data coming from the Cech nerve of this cover. We will also need the following variant.

Definition 4.5. (i) Let (A, I) be a bounded prism, we define the category of *analytic prismatic F-crystals* over (A, I), denoted by $\mathrm{Vect}^{\mathrm{an},\varphi}(A, I)$, to be the category of pairs (E, φ_E) where E is a vector bundle over $\mathrm{Spec}(A)\backslash V(p, I)$, and φ_E is an isomorphism $\varphi_A^* E[1/I] \xrightarrow{\sim} E[1/I]$. Morphisms in $\mathrm{Vect}^{\mathrm{an},\varphi}(A, I)$ are morphisms between vector bundles that are compatible with Frobenius. [2]

(ii) Let $\mathfrak{X}/\mathcal{O}_K$ be a smooth p-adic formal scheme. We define the category $\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ of *analytic prismatic F-crystals* by the derived limit

$$\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}) := \lim_{(A,I)\in\mathfrak{X}_{\Delta}} \mathrm{Vect}^{\mathrm{an},\varphi}(A, I).$$

We have a natural forgetful functor $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \longrightarrow \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ from prismatic F-crystals to analytic prismatic F-crystals. This functor is induced from the map $\mathrm{Vect}^{\varphi}(A, I) \to \mathrm{Vect}^{\mathrm{an},\varphi}(A, I)$ that sends a vector bundle over $\mathrm{Spec}\, A$ to its restriction over the complement of $V(p, I)$ for each $(A, I) \in \mathfrak{X}_{\Delta}$ and is fully faithful. Moreover, it is compatible with crystalline realizations, in the sense that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \arrow[r] \arrow[d] & \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}) \arrow[d] \\ \mathrm{Vect}^{\varphi}(\mathfrak{X}_{s,\mathrm{crys}}) \arrow[r] & \mathrm{Isoc}^{\varphi}(\mathfrak{X}_{s,\mathrm{crys}}) \end{tikzcd} \tag{2}$$

where the vertical arrows are induced by specializing to the special fiber $\mathfrak{X}_s$ of $\mathfrak{X}$ and identifying the absolute prismatic site $\mathfrak{X}_{s,\Delta}$ of $\mathfrak{X}_s$ with the p-completed absolute crystalline site $\mathfrak{X}_{s,\mathrm{crys}}$ of $\mathfrak{X}_s$ (see [BS23, Construction 4.12]). Let us consider a special case of this restriction functor in the setting of Example 4.2.

Lemma 4.6. *Let $\mathfrak{X} = \mathrm{Spf}\, R$, where $R = (\mathcal{O}_K[t])_p^\wedge$. Let $U = \mathrm{Spec}(\mathfrak{S}_R)\backslash V(p, E(u))$ be the open subset of $\mathrm{Spec}(\mathfrak{S}_R)$ and denote by $j : U \to \mathrm{Spec}(\mathfrak{S}_R)$ the open immersion. The essential image of the fully faithful functor*

$$\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \longrightarrow \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$$

consists of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$ satisfying the following condition: if we write (E_U, φ_U) for the vector bundle over U obtained by evaluating $(\mathcal{E}, \varphi_{\mathcal{E}})$ on the prism $(\mathfrak{S}_R, E(u)) \in \mathfrak{X}_{\Delta}$, then $j_* E_U$ is a vector bundle over $\mathrm{Spec}\, \mathfrak{S}_R$.

[2] Note that, for a prism (A, I), the Frobenius φ_A preserves the zero locus $V(p, I) \subset \mathrm{Spec}\, A$ as well as its complement. Thus the definition above makes sense.

Proof. This is [IKY24, Proposition 1.26]. □

Definition 4.7. Let $\mathfrak{X}/\mathcal{O}_K$ be a smooth p-adic formal scheme. Let $\mathfrak{X}_\eta/K$ be its adic generic fiber and let $\mathfrak{X}_s/k$ denote the special fiber. Let $\mathbb{L}$ be an étale $\mathbb{Z}_p$-local system on $\mathfrak{X}_\eta$. We say that $\mathbb{L}$ is *crystalline* if there exists an F-isocrystal $(\mathbb{E}, \varphi)$ over $\mathfrak{X}_s$, together with a Frobenius equivariant isomorphism of $\mathbb{B}_{\mathrm{crys}}$-vector bundles

$$\mathbb{B}_{\mathrm{crys}}(\mathbb{E}) \xrightarrow{\sim} \mathbb{B}_{\mathrm{crys}} \otimes_{\mathbb{Z}_p} \mathbb{L}.$$

Here $\mathbb{B}_{\mathrm{crys}}(\mathbb{E})$ is the sheaf of $\mathbb{B}_{\mathrm{crys}}$-modules on the pro-étale site $\mathfrak{X}_{\eta,\text{proét}}$ associated to $(\mathbb{E}, \varphi)$. We denote the category of crystalline $\mathbb{Z}_p$-local systems on $\mathfrak{X}_\eta$ by $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$. [3]

We shall need the following result in the article, due to Bhatt–Scholze [BS23] in the case of Spf $\mathcal{O}_K$ and to Du–Liu–Moon–Shimizu [DLMS24]/Guo–Reinecke [GR24] in general.

Theorem 4.8 ([BS23, DLMS24, GR24])**.** *Let $\mathfrak{X}/\mathcal{O}_K$ be a smooth p-adic formal scheme. There is a natural equivalence of categories*

$$\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta).$$

In particular, this equivalence is functorial in $\mathfrak{X}$.

Remark 4.9. In the setting above, if $\mathbb{L}$ is a crystalline $\mathbb{Z}_p$-local system on $\mathfrak{X}_\eta$, then one can recover the F-isocrystal over the special fiber $\mathfrak{X}_s$ from this equivalence via the crystalline realization (see Diagram (2)).

Example 4.10. Let us revisit Example 4.2 once again.

(i) Let $(\mathfrak{S}, E)$ be the Breuil–Kisin prism, and let $U_0 = \mathrm{Spec}(\mathfrak{S})\backslash V(p, E)$ denote the open subscheme obtained as the complement of a closed point in $\mathrm{Spec}\,\mathfrak{S}$. Theorem 4.8 (due to Bhatt–Scholze in this context) says that a $\mathbb{Z}_p$-lattice Λ in a crystalline Gal_K-representation is equivalent to the data of a vector bundle $\mathcal{E}_0$ over $U_0 = \mathrm{Spec}(\mathfrak{S})\backslash V(p, E)$, equipped with a Frobenius isomorphism after inverting E, as well as the descent data coming from the cosimplicial complex (1). Since $\mathfrak{S}$ is a regular local scheme of dimension 2, $\mathcal{E}_0$ uniquely extends to a vector bundle over $\mathrm{Spec}\,\mathfrak{S}$, which still carries the Frobenius isomorphism after inverting E, and again satisfies descent. In other words, for Spf $\mathcal{O}_K$, the fully faithful embedding $\mathrm{Vect}^{\varphi}((\mathrm{Spf}\,\mathcal{O}_K)_{\Delta}) \to \mathrm{Vect}^{\mathrm{an},\varphi}((\mathrm{Spf}\,\mathcal{O}_K)_{\Delta})$ is an equivalence. Moreover, via Theorem 4.8 and this equivalence, a $\mathbb{Z}_p$-lattice Λ in a crystalline representation is equivalent to a *Breuil–Kisin module* $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ that satisfies descent along the cosimplicial complex (1), that is, equipped with a descent isomorphism after pulling back to $\mathfrak{S}^{(1)}$ which satisfies a cocycle condition over $\mathfrak{S}^{(2)}$.

(ii) Let $R = (\mathcal{O}_K[t]^{\wedge}_p)$ be as in Example 4.2 (2) and let $\mathfrak{X} = \mathrm{Spf}\,R$, so $\mathfrak{X}_\eta = \mathsf{D} = \mathrm{Spa}(R[1/p], R)$ is the closed unit disc over K. In this case, Theorem 4.8 tells us that crystalline $\mathbb{Z}_p$-local system on D is equivalent to the category of analytic prismatic F-crystals over $\mathfrak{X}$. In particular, it gives rise to a vector bundle E_U over $U = \mathrm{Spec}(\mathfrak{S}_R)\backslash V(p, E(u))$ equipped with a Frobenius φ_E (by evaluating the analytic prismatic F-crystal on the prism $(\mathfrak{S}_R, E(u))$).

[3]Note that, *a priori*, the notion of crystallinity of local systems depends on the integral model $\mathfrak{X}$ over $\mathcal{O}_K$. In fact, this notion only only depends on the generic fiber $\mathfrak{X}_\eta$ (and independent of the chosen integral model).

5 From F-isocrystals to F-crystals

Let $\mathbb{L}$ be a crystalline $\mathbb{Z}_p$-local system on the closed unit disc D, and let $\mathbb{E}$ be the F-isocrystal over the special fiber of D (which is a copy of $\mathbb{A}^1_k$) attached to $\mathbb{L}$. Let us also recall that, if V is a crystalline $\mathbb{Q}_p$-representation of the Galois group Gal_K with nonnegative Hodge–Tate weights, and $\Lambda \subset V$ is a Gal_K-stable $\mathbb{Z}_p$-lattice, then one can attach to Λ a Frobenius module

$$\mathbb{D}_{\mathrm{crys}}(\Lambda) = (M, \varphi_M),$$

which consists of a finite free $W(k)$-module M together with a Frobenius map $\varphi_M : M \to M$ that becomes an isomorphism upon inverting p. This can be achieved by considering the Breuil–Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ and base changing along the map $\mathfrak{S} = W(k)[\![u]\!] \to W(k)$ sending $u \mapsto 0$. We may regard $\mathbb{D}_{\mathrm{crys}}(\Lambda)$ as an F-crystal over $\mathrm{Spec}\, k$ (see Example 4.10 and also see Diagram (2)). The goal of this section is to show the following.

Theorem 5.1. *Assume the above setup. Up to replacing D by a smaller closed disc in D if necessary, there exists an F-crystal $\mathbb{D}$ over $\mathbb{A}^1$ with $\mathbb{D}[1/p] \cong \mathbb{E}$, that is compatible with the crystalline $\mathbb{Z}_p$-local system $\mathbb{L}$ in the following sense: for every finite extension L/K and every classical L-point $x \in \mathsf{D}$ which specializes to a closed point $\overline{x} \in \mathbb{A}^1$, there is an isomorphism $\mathbb{D}|_{\overline{x}} \cong \mathbb{D}_{\mathrm{crys}}(\mathbb{L}|_x)$ of F-crystals over $\overline{x}$. Consequently, up to replacing D by a smaller disc, the isomorphism class of the F-crystal $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}_x)$ is independent of the classical point $x \in \mathsf{D}$.*

5.1 Locally free extensions

Let $\mathcal{T}$ be an admissible p-adic formal scheme over $\mathrm{Spf}\, W$, and consider the sheaf of rings $\mathcal{O}_{\mathcal{T}}[\![u]\!]$.[4] We say $\mathcal{T}$ is regular if every local ring is regular. In this section we prove the following:

Proposition 5.2. *Let $\mathcal{T}$ be a regular 2-dimensional p-adic formal scheme over $\mathrm{Spf}\, W$ and E a finitely generated $\mathcal{O}_{\mathcal{T}}[\![u]\!]$-module. Then there is a sequence*

$$\mathcal{T}' = \mathcal{T}_n \to \cdots \to \mathcal{T}_1 = \mathcal{T}$$

of formal admissible blow-ups at closed points, such that $(f[\![u]\!]^*E)^{\vee\vee}$ is locally free as an $\mathcal{O}_{\mathcal{T}}[\![u]\!]$-module, where $f : \mathcal{T}' \to \mathcal{T}$ denotes the composition of sequence of maps above.

We begin with the following observation.

Lemma 5.3. *Let $\mathcal{O}$ be a 3-dimensional regular local ring and E a finitely generated reflexive $\mathcal{O}$-module. Then $\mathrm{Ext}^i_{\mathcal{O}}(E, \mathcal{O}) = 0$ for $i > 1$.*

Proof. Every term in the double dual spectral sequence

$$R\,\mathrm{Hom}_{\mathcal{O}}(R\,\mathrm{Hom}_{\mathcal{O}}(E, \mathcal{O}), \mathcal{O}) = E$$

vanishes except $E^{\vee\vee}$, $\mathrm{Ext}^1_{\mathcal{O}}(E^\vee, \mathcal{O})$, and $\mathrm{Ext}^3_{\mathcal{O}}(\mathrm{Ext}^1_{\mathcal{O}}(E, \mathcal{O}), \mathcal{O})$. In particular, this implies that $\mathrm{Ext}^3_{\mathcal{O}}(\mathrm{Ext}^i_{\mathcal{O}}(E, \mathcal{O}), \mathcal{O}) = 0$ for $i > 1$, so $\mathrm{Ext}^i_{\mathcal{O}}(E, \mathcal{O}) = 0$ by local duality. □

The main step of Proposition 5.2 is the following:

[4] In the context of Example 4.2(2), the reader may take $\mathcal{T} = \mathrm{Spf}\, R_0$ as a working example in this subsection.

Lemma 5.4. *Let $\mathcal{O}$ be a 3-dimensional regular local ring and E a finitely generated reflexive $\mathcal{O}$-module which is not free. Let $f : X \to \operatorname{Spec}\mathcal{O}$ be the blow-up along any regular curve. Then we have*

$$\ell(\mathcal{E}xt^1_{\mathcal{O}_X}((f^*E)^\vee, \mathcal{O}_X)) < \ell(\operatorname{Ext}^1_{\mathcal{O}}(E^\vee, \mathcal{O})),$$

where ℓ denotes the length of ($\mathcal{O}$-)modules.

Proof. Step 1. E has a presentation of the form

$$0 \longrightarrow \mathcal{O}^m \xrightarrow{A} \mathcal{O}^n \longrightarrow E \longrightarrow 0 \ .$$

Proof. Consider any presentation

$$\mathcal{O}^m \xrightarrow{A} \mathcal{O}^n \longrightarrow E \longrightarrow 0 \ .$$

We claim that $N = \operatorname{img}(A)$ is free. Indeed, applying $RHom_{\mathcal{O}}(-, \mathcal{O})$ to the sequence

$$0 \longrightarrow N \longrightarrow \mathcal{O}^n \longrightarrow E \longrightarrow 0$$

and using Lemma 5.3, we see that $\operatorname{Ext}^i_{\mathcal{O}}(N, \mathcal{O}) = 0$ for $i > 0$. ☐

Step 2. Let M be any nonzero finitely-generated $\mathcal{O}$-module supported on (some thickening of) the closed point x of $\operatorname{Spec}\mathcal{O}$. Then $L^i f^* M = 0$ for $i < -1$ and $H^0(\mathcal{E}xt^2_{\mathcal{O}_X}(L^{-1}f^*M, \mathcal{O}_X)) \neq 0$.

Proof. First observe that the vanishing claim is true for $M = k(x)$ using the Koszul resolution. In general we may take a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow k(x) \longrightarrow 0.$$

and the vanishing follows from the above observation by induction on $\ell(M)$.

For the second claim, again observe that the claim is true for $M = k(x)$ since $L^{-1}f^*k(x) = \mathcal{O}_C(-1)$ where $C = f^{-1}(x) \cong \mathbb{P}^1_{k(x)}$ and $\mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{O}_C(-1), \mathcal{O}_X) = \mathcal{O}_C$. In general, using the same sequence we have an exact sequence

$$0 \longrightarrow L^{-1}f^*M' \longrightarrow L^{-1}f^*M \longrightarrow \mathcal{O}_C(-1)$$

and the image of the rightmost map is therefore either 0 or $\mathcal{O}_C(-a)$ for some $a \geq 1$. Thus by induction we may assume we are in the latter case. But then we have an inclusion

$$0 \longrightarrow \mathcal{O}_C(a-1) \cong \mathcal{E}xt^2_{\mathcal{O}_X}(\mathcal{O}_C(-a), \mathcal{O}_X) \longrightarrow \mathcal{E}xt^2_{\mathcal{O}_X}(L^{-1}f^*M, \mathcal{O}_X)$$

whence the claim. ☐

Taking the dual of the presentation from Step 1 we have

$$0 \longrightarrow E^\vee \longrightarrow \mathcal{O}^n \xrightarrow{A^*} \mathcal{O}^m \longrightarrow \operatorname{Ext}^1_{\mathcal{O}}(E, \mathcal{O}) \longrightarrow 0.$$

Let $F = \operatorname{img}(A^*)$ and $Q = \operatorname{Ext}^1_{\mathcal{O}}(E, \mathcal{O})$.

Step 3. We have $L^i f^* F = 0$ and $L^i f^*(E^\vee) = 0$ for $i < 0$.

Proof. Pulling back the sequence

$$0 \longrightarrow F \longrightarrow \mathcal{O}^m \longrightarrow Q \longrightarrow 0$$

and using the vanishing in Step 2 implies the first claim. Pulling back

$$0 \longrightarrow E^\vee \longrightarrow \mathcal{O}^n \longrightarrow F \longrightarrow 0$$

and using the first claim implies the second. □

Step 4. There is a natural short exact sequence

$$0 \longrightarrow f^*(E^\vee) \longrightarrow (f^*E)^\vee \longrightarrow L^{-1}f^*Q \longrightarrow 0.$$

Proof. Since $\mathcal{O}_X$ is torsion-free, pulling back the presentation from Step 1 we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X^m \xrightarrow{f^*A} \mathcal{O}_X^n \longrightarrow f^*E \longrightarrow 0$$

and therefore an exact seqeunce

$$0 \longrightarrow (f^*E)^\vee \longrightarrow \mathcal{O}_X^n \xrightarrow{f^*A^*} \mathcal{O}_X^m.$$

Now applying the previous step to the diagram with exact diagonals

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & & \\
 & & L^{-1}f^*Q \arrow[dr] & & 0 & & \\
 & & & f^*F \arrow[ur] \arrow[dr] & & & \\
 & & \mathcal{O}_X^n \arrow[ur] \arrow[rr, "f^*A^*"'] & & \mathcal{O}_X^m \arrow[dr] & & \\
 & f^*(E^\vee) \arrow[ur] & & & & f^*Q \arrow[dr] & \\
L^{-1}f^*F \arrow[ur] & & & & & & 0
\end{tikzcd}$$

yields the claim. □

Step 5. There is a natural short exact sequence

$$0 \longrightarrow H^1(f^*(E^\vee)^\vee) \longrightarrow \operatorname{Ext}^1_{\mathcal{O}}(E^\vee, \mathcal{O}) \longrightarrow H^0(\mathcal{E}\mathrm{xt}\,^1_{\mathcal{O}_X}(f^*(E^\vee), \mathcal{O}_X)) \longrightarrow 0.$$

In particular, $\ell(\mathcal{E}\mathrm{xt}\,^1_{\mathcal{O}_X}(f^*(E^\vee), \mathcal{O}_X)) \le \ell(\operatorname{Ext}^1_{\mathcal{O}}(E^\vee, \mathcal{O}))$.

Proof. Use

$$Rf_*(R\mathcal{H}\mathrm{om}\,_{\mathcal{O}_X}(Lf^*(E^\vee), \mathcal{O}_X)) = R\operatorname{Hom}_{\mathcal{O}}(E^\vee, Rf_*\mathcal{O}_X)$$

together with $Rf_*\mathcal{O}_X = \mathcal{O}$ and the vanishing of $L^if^*(E^\vee)$ for $i < 0$ from Step 3. □

Step 6.

$$\ell(\mathcal{E}\mathrm{xt}\,^1_{\mathcal{O}_X}((f^*E)^\vee, \mathcal{O}_X)) \le \ell(\operatorname{Ext}^1_{\mathcal{O}}(E^\vee, \mathcal{O})) - \ell(H^0(\mathcal{E}\mathrm{xt}\,^2_{\mathcal{O}_X}(L^{-1}f^*Q, \mathcal{O}_X))).$$

Proof. Applying $R\operatorname{Hom}_{\mathcal{O}_X}(-,\mathcal{O}_X)$ to the sequence from Step 4 and using the vanishing from Lemma 5.3 we have an exact sequence

$$0 \longrightarrow \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}((f^*E)^\vee,\mathcal{O}_X) \longrightarrow \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(f^*(E^\vee),\mathcal{O}_X) \longrightarrow \mathcal{E}\mathrm{xt}^2_{\mathcal{O}_X}(L^{-1}f^*Q,\mathcal{O}_X) \longrightarrow 0.$$

The first term has dimension 0, so the sequence remains exact on taking global sections

$$0 \to H^0(\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}((f^*E)^\vee,\mathcal{O}_X)) \to H^0(\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_X}(f^*(E^\vee),\mathcal{O}_X)) \to H^0(\mathcal{E}\mathrm{xt}^2_{\mathcal{O}_X}(L^{-1}f^*Q,\mathcal{O}_X)) \to 0.$$

Combining this with Step 5, we have the claim. □

Step 7. Conclusion of proof.

By the assumption on E and Lemma 5.3 we have $Q \neq 0$. But then by the nonvanishing in Step 2 we have $0 \neq \ell(H^0(\mathcal{E}\mathrm{xt}^2_{\mathcal{O}_X}(L^{-1}f^*Q,\mathcal{O}_X)))$, so the claim follows from Step 6. □

Proof of Proposition 5.2. Note it is equivalent to show $(f[\![u]\!]^*E)^\vee$ is locally free. Since $E^\vee$ is a reflexive $\mathcal{O}_{\mathcal{T}}[\![u]\!]$-module, we have

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_{\mathcal{T}}[\![u]\!]}(E^\vee,\mathcal{O}_{\mathcal{T}}[\![u]\!]) = 0$$

for $i > 1$ by Lemma 5.3. Moreover, $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{\mathcal{T}}[\![u]\!]}(E^\vee,\mathcal{O}_{\mathcal{T}}[\![u]\!])$ is supported at finitely many closed points, hence has finite length ℓ. We proceed by induction on ℓ, the $\ell = 0$ case being trivial. Let $x \in \mathcal{T}$ be any point in the support, $\mathcal{O}_{\mathcal{T},x}$ the local ring of $\mathcal{T}$ at x with maximal idea $\mathfrak{m}_x$, and let $\mathfrak{n}_x$ be the ideal generated by $\mathfrak{m}_x$ in $\mathcal{O}_{\mathcal{T},x}[\![u]\!]$. Let

$$\pi : B \to \operatorname{Spec}\mathcal{O}_{\mathcal{T},x}$$

be the blow-up at $\mathfrak{m}_x$, and note that:

- the base-change $\pi[\![u]\!] : B[\![u]\!] \to \operatorname{Spec}(\mathcal{O}_{\mathcal{T},x}[\![u]\!])$ is the blow-up at $\mathfrak{n}_x$, where $B[\![u]\!]$ denotes the scheme theoretic base change $B[\![u]\!] := B \times_{\operatorname{Spec}(\mathcal{O}_{\mathcal{T},x}} \operatorname{Spec}(\mathcal{O}_{\mathcal{T},x}[\![u]\!])$;
- the p-adic completions of the local rings of B are identified with the local rings of the formal admissible blow-up
$$g : \mathcal{T}' \to \mathcal{T}$$
of $\mathcal{T}$ at x.

In particular, for a closed point x' of $\mathcal{T}'$ (which we also view as a closed point in $B[\![u]\!]$), the p-adic complete local ring

$$(\mathcal{O}_{\mathcal{T}'}[\![u]\!])_{x'} = \mathcal{O}_{\mathcal{T}',x'}[\![u]\!]$$

is flat over $\mathcal{O}_{B[\![u]\!],x'}$. Here $(\mathcal{O}_{\mathcal{T}'}[\![u]\!])_{x'}$ denotes the stalk of the sheaf $\mathcal{O}_{\mathcal{T}'}[\![u]\!]$ of the p-adic formal scheme $\mathcal{T}'$ at x'

According to Lemma 5.4, the length of $\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{B[\![u]\!],x'}}((\pi[\![u]\!]^*E_x)^\vee,\mathcal{O}_{B[\![u]\!],x'})$ is strictly smaller than

$$\ell_x := \mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{\mathcal{T},x}[\![u]\!]}(E_x^\vee,\mathcal{O}_{\mathcal{T},x}[\![u]\!]),$$

so the same is true of $\mathcal{E}\mathrm{xt}^1_{(\mathcal{O}_{\mathcal{T}'}[\![u]\!])_{x'}}((g[\![u]\!]^*E_x)^\vee,(\mathcal{O}_{\mathcal{T}'}[\![u]\!])_{x'})$. Therefore, we know that

$$\ell\big(\mathcal{E}\mathrm{xt}^1_{\mathcal{O}_{\mathcal{T}'}[\![u]\!]}((g[\![u]\!]^*E)^\vee,\mathcal{O}_{\mathcal{T}'}[\![u]\!])\big) < \ell.$$

By induction the proof is complete. □

5.2 Existence of the Prismatic F-crystal

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. Let $\mathfrak{X} = \operatorname{Spf} R = \operatorname{Spf}(\mathcal{O}_K[t])_p^\wedge$ be an integral model of D as in the setting of Example 4.10 (2). In particular, the crystalline local system $\mathbb{L}$ gives rise to a vector bundle E_U over $U = \operatorname{Spec}(\mathfrak{S}_R)\backslash V(p, E(u))$, where we recall that $\mathfrak{S}_R = R_0[\![u]\!]$ and $R_0 = (W[t_0])_p^\wedge$. Let $\mathfrak{X}_0 = \operatorname{Spf} R_0$ (which is another copy of the p-adic formal $\mathbb{A}^1$). Let $E = j_* E_U$ where $j : U \to \operatorname{Spec} \mathfrak{S}_R$ is the open immersion, which is a reflexive module over $\mathcal{O}_{\mathfrak{X}_0}[\![u]\!]$. By Proposition 5.2, we know that there exists successive admissible blowups of $\mathfrak{X}_0$ at closed points such that the pullback $\pi[\![u]\!]^* E$ is a locally free $\mathcal{O}_{\mathfrak{X}'_0}[\![u]\!]$-module, where $\pi : \mathfrak{X}'_0 \to \mathfrak{X}_0$ is the composition of the required blowups. From the proof of Proposition 5.2, we know that by shrinking the radius of the disc in the generic fiber at each step of the blowup if necessary, we may without loss of generality assume that at each step we only need to blowup at the origin. In other words, the required blowups π can be achieved by the map $\pi_n : R_0 = (W[t_0])_p^\wedge \to R_n = (W[t_n])_p^\wedge$ sending $t_0 \mapsto p^n t_n$ for some large enough integer n. This means that, after replacing the disc D by its closed subdisc $\mathsf{D}_n = \{|z| \leq 1/p^n\}$, the local system $\mathbb{L}|_{\mathsf{D}_n}$ gives rise to a pair $(E_{n,U}, \varphi_{n,U})$ where $E_{n,U}$ is a vector bundle over $U_n = \operatorname{Spec} R_n[\![u]\!]\backslash V(p, E(u))$ that extends to a vector bundle over $\operatorname{Spec} R_n[\![u]\!]$. By Lemma 4.6, $\mathbb{L}|_{\mathsf{D}_n}$ in fact gives rise to a (unique up to isomorphism) prismatic F-crystal $(\mathcal{E}_n, \varphi_n)$ over $\operatorname{Spf}(\mathcal{O}_K[t'])_p^\wedge$ where $t' = p^n t$. This prismatic F-crystal is compatible with the F-crystal $\mathbb{D}_{\mathrm{crys}}(\mathbb{L}|_x)$ under specialization to closed points $x \in \mathsf{D}_n$ by the functoriality of the equivalence in Theorem 4.8. This finishes the proof of the theorem. $\square$

6 Constancy of the F-crystal

In this section, we will prove that the fiber-wise F-crystal produced in Section 5 is actually constant. Our argument is essentially Lemmas 1.10 and 1.9 of [Oor04] in the setting of F-crystals.

Lemma 6.1 (Oort, Lemma 1.10)**.** *Let $\mathbb{D}_1$ and $\mathbb{D}_2$ be two F-crystals over $\overline{\mathbb{F}}_p$ having the same rank and let N be a fixed integer. Then there are finitely many sub-crystals of $\mathbb{D}_2$ isomorphic to $\mathbb{D}_1$ whose co-kernel is p^N torsion.*

Proof. Let $L = \operatorname{Hom}(\mathbb{D}_1, \mathbb{D}_2)$. Consider the set $S \subset L$ defined as $S = \{f \in L : \operatorname{Im}(f) \supset p^N \mathbb{D}_2\}$. Let $f_1, f_2 \in S$ be two maps such that $f_1 \equiv f_2 \mod p^N$ in L/p^N. A direct computation shows that $\operatorname{Im} f_1 = \operatorname{Im} f_2$. It follows that the number of sub-crystals is bounded by the image of the subset S in L/p^N. The lemma follows from the fact that L is a finite generated $\mathbb{Z}_p$ module. $\square$

Definition 6.2. Let $X/\overline{\mathbb{F}}_p$ be a scheme and $\mathbb{D}$ an F-(iso)crystal. Let $\mathrm{pt}_X : X \to \operatorname{Spec} \overline{\mathbb{F}}_p$ be the map to a point, and for any point $x \in X(\overline{\mathbb{F}}_p)$ let $i_x : \operatorname{Spec} \overline{\mathbb{F}}_p \to X$ be the corresponding morphism.

1. We say $\mathbb{D}$ is *constant* if there is an (iso)crystal $\mathbb{D}_0$ on $\operatorname{Spec} \overline{\mathbb{F}}_p$ such that if $\mathbb{D} \cong \mathrm{pt}_X^* \mathbb{D}_0$.
2. We say $\mathbb{D}$ is *point-wise constant* if there is an (iso)crystal $\mathbb{D}_0$ on $\operatorname{Spec} \overline{\mathbb{F}}_p$ such that $i_x^* \mathbb{D} \cong \mathbb{D}_0$ for every $x \in X(\overline{\mathbb{F}}_p)$.

We are now ready to prove

Theorem 6.3. *Let X be a smooth connected variety over $\overline{\mathbb{F}}_p$. Let $\mathbb{D}/X$ be a point-wise constant F-crystal such that the F-isocrystal $\mathbb{D}[1/p]$ is constant. Then $\mathbb{D}$ itself is constant.*

Proof. Let $\mathbb{D}_0$ be the F-crystal $i_x^*\mathbb{D}$ for $x \in X(\overline{\mathbb{F}}_p)$. The constancy of $\mathbb{D}[1/p]$ gives an isogeny $g : \mathbb{D} \to \mathrm{pt}_X^* \mathbb{D}_0$. By Lemma 6.1, the set $X(\overline{\mathbb{F}}_p)$ admits a partition into finitely many subsets $X_1 \ldots X_m$ such that for two points $x, x' \in X_i$, we have $g_x(\mathbb{D}_x) = g_{x'}(\mathbb{D}_{x'})$. Without loss of generality, suppose that X_1 is Zariski dense, and let $x \in X_1$ be any point. Consider the inclusion of crystals $g' : \mathrm{pt}_X^* \mathbb{D}_0 \to \mathrm{pt}_X^* \mathbb{D}_0$, where $g' = \mathrm{pt}_X^* g_x$. We now have that $\operatorname{Im} g'$ and $\operatorname{Im} g$ are two subcrystals of $\mathrm{pt}^* \mathbb{D}_0$ which agree on a Zariski-dense set of points. We will now show that they are the same.

We first reduce to the case that X is affine. We claim that for two crystals $\mathbb{D}_1, \mathbb{D}_2$ on X and any dense Zariski open $U \subset X$, any morphism $f_U : \mathbb{D}_1|_U \to \mathbb{D}_2|_U$ extends uniquely to a morphism $f : \mathbb{D}_1 \to \mathbb{D}_2$. By the uniqueness, such an extension patches, so we may assume X is affine, and in particular lifts to a formally smooth scheme $\mathfrak{X}/W(\overline{\mathbb{F}}_p)$ equipped with a lift of Frobenius $\phi : \mathfrak{X} \to \mathfrak{X}$. Then $\mathbb{D}_1$ corresponds to a flat vector bundle E_1 on $\mathfrak{X}$ together with a flat morphism $\phi_{E_1} : \phi^* E_1 \to E_1$ for which $\phi_{E_1}[1/p]$ is an isomorphism; likewise for $\mathbb{D}_2$ and E_2. Then f yields a morphism $\mathfrak{f}_{\mathfrak{U}} : E_1|_{\mathfrak{U}} \to E_2|_{\mathfrak{U}}$. In characteristic 0, it is easy to see that $\mathfrak{f}_{\mathfrak{U}}[1/p]$ extends uniquely to a moprhism $E_1|_{\mathfrak{X}[1/p]} \to E_2|_{\mathfrak{X}[1/p]}$. But then the underlying morphism of vector bundles $E_1 \to E_2$ is defined outside of a set of codimension 2, hence extends uniquely, and the extension is compatible with the connection and the Frobenius structure.

Therefore, we suppose that $X = \operatorname{Spec} A$. Let $\tilde{A}$ (cite the people Kedlaya cites!) be a p-adically complete smooth W-algebra with a lift of Frobenius such that $A = \tilde{A} \mod p$. Evaluating all these crystals on $\tilde{A}$, we obtain two Frobenius-stable flat-subbundles of $\mathbb{D}_0 \otimes \tilde{A}$ which:

1. Are the same after inverting p.

2. Agree at a dense set of W-points.

Even without using F-crystal structure, the formal-smoothness of $\tilde{A}$ implies that the these two sub-bundles must already agree. The theorem follows.

□

7 Proof of the extension theorem for $\mathsf{D}^\times$

In this section, we prove our main results for $\mathsf{D}^\times$.

7.1 Extending crystalline local systems

The following result is Theorem 1.17 of [DY25].

Theorem 7.1 ([DY25])**.** *Let $\mathbb{L}/\mathsf{D}^\times$ be a $\mathbb{Z}_p$-local system with crystalline fibers everywhere. Then, $\mathbb{L}$ extends to a crystalline local system on D.*

For the sake of completeness, we will sketch a proof in the easy case where in addition there exists a smooth formal scheme $\mathfrak{X}$, a crystalline $\mathbb{Z}_p$-local system $\mathbb{L}'/\mathfrak{X}_\eta$, and a map $g : \mathsf{D}^\times \to \mathfrak{X}_\eta$ such that $\mathbb{L} = g^*\mathbb{L}'$. Let $\mathbb{E}[1/p]$ denote the isocrystal associated to $\mathbb{L}'$ on $\mathfrak{X}_p$. Consider the flat bundle underlying the filtered flat bundle $\mathcal{RH}_p(\mathbb{L})/\mathsf{D}^\times$ (resp. $\mathcal{RH}_p(\mathbb{L}')/\mathfrak{X}_\eta$)—we will somewhat abusively use the same notation both for the filtered flat bundles and the underlying flat bundles. We have that $\mathbb{E}[1/p](\mathfrak{X})$ is canonically isomorphic to $\mathcal{RH}_p(\mathbb{L}')$ (see for eg [BST24, Lemma 3.7]). It suffices to prove that $\mathcal{RH}_p(\mathbb{L})$ extends to a flat bundle on D, by [OSZP24, Theorem 5.7]. We will in fact use the fact that $\mathbb{L}$ is pulled back from $\mathfrak{X}_\eta$ to conclude that $\mathcal{RH}_p(\mathbb{L})|_{\mathsf{D}'}$ is actually the trivial flat bundle for every closed sub-disc[5] of $\mathsf{D}' \subsetneq \mathsf{D}$. Indeed, it suffices to prove that $\mathcal{RH}_p(\mathbb{L})|_{\mathsf{A}'_n}$ is the trivial flat bundle for every integer n, where $\mathsf{A}'_n := \mathsf{A}_n \cap \mathsf{D}'$. We pick an integral model $\mathfrak{g}'_n : \mathfrak{A}'_n \to \mathfrak{A}_{n+1} \to \mathfrak{X}$ of the composite map $g'_n := \mathsf{A}'_n \subsetneq \mathsf{A}_n \xrightarrow{g|_{\mathsf{A}_{n+1}}} \mathfrak{X}_\eta$. By the appendix of [OSZP24], we may assume that the irreducible components of the reduced special fiber C_{n+1} (resp. C'_n) of $\mathfrak{A}_{n+1}$ (resp. $\mathfrak{A}'_n$) consist of (two) $\mathbb{A}^1$s and (an unspecified but finite number of) $\mathbb{P}^1$s, with the further condition that the there is no combinatorial monodromy. By [OSZP24, Remark A.7], there is a marked point x_i on each of the two $\mathbb{A}^1$s in C_{n+1} such that no point in the open annulus A°_{n+1} specializes to a point in $\mathbb{A}^1 \setminus \{x_i\}$. It follows that the image of C'_n in C_{n+1} is contained in a proper (possibly reducible) curve, and in particular, $\mathfrak{g}'_n$ extends to the compactification $\bar{C}'_n$ of C'_n, whose irreducible components are now solely $\mathbb{P}^1$s (and there is still no combinatorial monodromy). Every F-isocrystal on $\bar{C}'_n$ is trivial, and therefore $\mathfrak{g}'^{,*}_n(\mathbb{E}[1/p])$ is trivial. Evaluating on the thickening given by $\mathfrak{A}'_n$, we have that $\mathfrak{g}'^{,*}_n(\mathbb{E}[1/p])(\mathfrak{A}'_n)$ is trivial. The result now follows from the chain of isomorphisms $\mathfrak{g}'^{,*}_n(\mathbb{E}[1/p])(\mathfrak{A}'_n) \simeq g'^{,*}_n(\mathbb{E}[1/p](\mathfrak{X})) \simeq g'^{,*}_n(\mathcal{RH}_p(\mathbb{L})) \simeq \mathcal{RH}_p(\mathbb{L}|_{\mathsf{A}'_n})$.

A very similar argument also works in the setting of geometric period images. Indeed, let j be the smallest integer such that the image of A_n is contained in $(Y^j)^{\mathrm{an}}$. Then, there is a finite subset $\Xi \subset \mathsf{A}_n$ whose complement maps to $(Y^j)^{\mathrm{an}} \setminus (Y^{j-1})^{\mathrm{an}}$. As $S^j \to Y^j$ is an isomorphism away from Y^{j-1}, $\mathsf{A}_n \setminus \Xi$ lifts to $(S^j)^{\mathrm{an}}$. The fact that $S^j \to Y^j$ is proper allows us to extend this to a map $\mathsf{A}_n \to (S^j)^{\mathrm{an}}$. Now, the identical argument outlined above works.

7.2 The Shimura case

We have the following proposition.

Proposition 7.2. *Let $g : C \to \mathscr{S}_p$ be a map where C is a curve over $\overline{\mathbb{F}}_p$. Suppose that $g^*\mathbb{V}_{\mathrm{cris}}$ is constant on C. Then the map g is constant.*

Proof. We first replace C by its image – note that the F-crystal remains constant. By replacing C by an open subset, we may assume that C is a smooth curve. We pick a smooth lift $\tilde{C}/W$. Consider $g^*(\mathcal{V}, \nabla)_{\mathrm{dR}}$, the filtered flat bundle on S pulled back to C. The underlying flat bundle is obtained by taking the mod p reduction of the flat bundle $(g^*\mathbb{V}_{\mathrm{cris}})(\tilde{C})$. The filtration is simply the kernel of (powers of) Frobenius mod p. Therefore, we have that $g^*(\mathcal{V}, \nabla)_{\mathrm{dR}}$ is constant as a filtered flat bundle. This implies that the Kodaira-Spencer map is trivial on C, which contradicts the versality of the Kodaira-Spencer map. The proposition follows. ☐

We are now ready to prove the one-dimensional disk case of Theorem 1.1 for a Shimura variety.

Theorem 7.3. *Theorem 1.1 is true for $a = 1, b = 0$ and X a Shimura variety.*

Proof. We will prove that the image of $\mathsf{D}^\times$ lands in a residue disc after suitably shrinking D – this would yield the theorem. The proof consists of assembling the various results already proved.

We have that $f^*\mathbb{L}$ extends to a local system with crystalline fibers on D. By Proposition 5.1, we may shrink D and obtain an F-crystal $\mathbb{D}$ on $\mathbb{A}^1 \bmod p$ such that for any classical $y \in \mathsf{D}$, $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}_y) = \mathbb{D}_{\bar{y}}$. By further shrinking D, we get that the F-crystal $\mathbb{D}$ is constant, and therefore the isomorphism class of $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}_y)$ is independent of the point y.

Let $\mathsf{A}_n \subset \mathsf{D}$ be a closed annulus with outer radius 1 and inner radius $\frac{1}{p^n}$, and let $f_n : \mathfrak{A}_n \to \mathscr{S}$ be an integral model for the map $f|_{\mathsf{A}_n}$. By [OSZP24, Appendix], we have that the special fiber C

[5]In fact, we do not need to restrict to D' for this triviality if $\mathfrak{X}$ satisfies the property that every map $\mathbb{A}^1 \to \mathfrak{X}_p$ extends to a map $\mathbb{P}^1 \to \mathfrak{X}_p$. We note that $\mathscr{S}$ does satisfy this property.

of $\mathfrak{A}_n$ is a union of two affine lines and some number of projective lines with trivial combinatorial monodromy. It suffices to show that $f_n|_{C_i}$ is constant for every component C_i of C.

Let $\bar{y} \in C_i$ and let $y \in \mathfrak{A}_n^{\mathrm{rig}}(K) = \mathsf{A}_n$ be a point that specializes to $\bar{y}$. We have that ${}_{\mathrm{cris}}\mathbb{V}_{\bar{y}} \simeq \mathbb{D}_{\mathrm{cris}(\mathbb{L}_y)}$ and therefore we have that $f_n^*({}_{\mathrm{cris}}\mathbb{V})_{C_i}$ is point-wise constant. Let $\bar{C}_i$ be the compactification of C_i. The map $C_i \to \mathscr{S}_p$ extends to $\bar{C}_i \to \mathscr{S}_p$ the results of Section 3. Therefore, the F-crystal $f_n^*(\mathbb{V}_{\mathrm{cris}})|_{C_n}$ extends to an F-crystal on $\bar{C}_i$, and therefore the F-isocrystal must be constant – indeed every F-isocrystal on $\mathbb{P}^1$ is constant. We are now in the situation of a point-wise constant F-crystal such that the underlying F-isocrystal is constant. By Theorem 6.3, we have that $f_n^*(\mathbb{V}_{\mathrm{cris}})|_{C_n}$ is constant. The theorem follows by applying Proposition 7.2. □

7.3 Period images

We have the following result.

Proposition 7.4. *Fix some integer j and consider $\pi^j : S^j \to Y^j$. Let $y \in \mathcal{Y}^j(\overline{\mathbb{F}}_p)$ denote some point and let $\mathcal{F}_y \subset \mathcal{S}^j$ denote the fiber over y. Then we have that $\mathbb{U}^j_{\mathrm{cris}}|_{\mathcal{F}_y}$ is point-wise constant. Further, the isomorphism class of this crystal is $\mathbb{D}_{\mathrm{cris}}(\mathbb{V}_{\acute{e}t,p_{\tilde{y}}})$, where $\tilde{y} \in Y(\mathcal{O}_K)$ is any lift of y.*

Remark 7.5. Note that this proposition assigns a canonical crystal to every $y \in \mathcal{Y}(\overline{\mathbb{F}}_p)$.

Proof. Let $T(\mathcal{F}_y)$ denote the tube in $\mathcal{S}^j$ over $\mathcal{F}_y$. For any lift $\tilde{y}$ of y, let let $F_{\tilde{y}} \subset T(\mathcal{F}_y)$ denote the fiber of π^j over $\tilde{y}$. As the stratification is uniform, there exists a lift $\tilde{y}$ of y, such the specialization map $F_{\tilde{y}} \to \mathcal{F}_y$ is surjective. Let $\tilde{z} \in T(\mathcal{F}_y)$ be any point and let $z \in \mathcal{F}_y(\overline{\mathbb{F}}_p)$ denote its specialization. As $\mathbb{U}^j_{\mathrm{\acute{e}t},p}$ is crystalline, we have a canonical isomorphism of crystals $\mathbb{D}_{\mathrm{cris}}(\mathbb{U}^j_{\mathrm{\acute{e}t},p_z}) \to \mathbb{U}^j_{\mathrm{cris}z}$.

Let $z_i, z_2 \in \mathcal{F}_y$. Let $\tilde{z}_1$ and $\tilde{z}_2$ be points in $F_{\tilde{y}}$ which specialize to z_1 and z_2 respectively. By the above remark, it suffices to prove that $\mathbb{D}_{\mathrm{cris}}(\mathbb{U}^j_{\mathrm{\acute{e}t},p_{\tilde{z}_1}})$ is isomorphic to $\mathbb{D}_{\mathrm{cris}}(\mathbb{U}^j_{\mathrm{\acute{e}t},p_{\tilde{z}_2}})$. But this follows directly from the fact that $\pi^j(\tilde{z}_1) = \pi^j(\tilde{z}_2)$ and that $\mathbb{U}_{\mathrm{\acute{e}t},p}$ is simply the pull-back of $\mathbb{V}_{\mathrm{\acute{e}t},p}$ under π^j. The proof of the second part now follows from this and the isomorphism $\mathbb{D}_{\mathrm{cris}}(\mathbb{U}^j_{\mathrm{\acute{e}t},p_{\tilde{z}}}) \to \mathbb{U}^j_{\mathrm{cris}z}$.

□

We are now ready to prove the one-dimensional extension theorem in this setting.

Theorem 7.6. *Theorem 1.1 is true for $a = 1, b = 0$ and X a geometric period image.*

Proof. As in the Shimura case, we will prove that the image of $\mathsf{D}^\times$ lands in a residue disc after suitably shrinking D. Also as in the Shimura case, we may further shrink $\mathsf{D}^\times$ such that the isomorphism class of $\mathbb{D}_{\mathrm{cris}}(\mathbb{L}_y)$ is independent of the point $y \in \mathsf{D}^\times$. We let A_n, $\mathfrak{A}_n$, f_n, and C be as in the Shimura case. Pick some component C_0 of C isomorphic to $\mathbb{P}^1$. Suppose that j is the smallest integer such that $f_n(C_0) \subset \mathcal{Y}^j$. Then, we have that C_0 generically maps into $\mathcal{Y}^j \setminus \mathcal{Y}^{j-1}$ and therefore we may lift f_n to a map $g_n : C_0 \to \mathcal{S}^j$. By Proposition 7.4, we have that $g_n^*(\mathbb{U}^j_{\mathrm{cris}})$ is point-wise constant. As C_0 is isomorphic to to $\mathbb{P}^1$, we also have that the iso-crystal $g_n^*(\mathbb{U}^j_{\mathrm{cris}}[1/p])$ is constant. By Theorem 6.3, we have that $g_n^*(\mathbb{U}^j_{\mathrm{cris}})$ is constant. As in the Shimura case, it follows that $g_n^*(\mathbb{U}^j_{\mathrm{FL}})$ is trivial as a filtered flat bundle. Up to passing to a finite cover C_0' of C_0, we may lift the map g_n to a map $h_n : C_0' \to \mathcal{T}^j$. We therefore have that $h_n^*(q^{i*}((\mathcal{U}, \nabla)_{\mathrm{dR}}{}^j))$ is trivial as a filtered flat bundle, as $(\mathcal{U}, \nabla)_{\mathrm{dR}}{}^j$ and $\mathbb{U}^j_{\mathrm{FL}}$ pull back to isomorphic filtered flat bundles on $\mathcal{T}^j$. Therefore, the Griffiths bundle is trivial on C_0'. If $f_n|_{C_0}$ is non-constant, we also have that Griffiths bundle on C_0 associated to $g_n^*((\mathcal{U}, \nabla)_{\mathrm{dR}}{}^j) \simeq f_n^*((\mathcal{V}, \nabla)_{\mathrm{dR}})$ is ample, and therefore the Griffiths bundle must be ample on C_0'. This is a contradiction. Therefore, $f_n|_{C_0}$ must be constant. It follows that the map f_n contracts every component isomorphic to $\mathbb{P}^1$. By [Bor72, Appendix Remark A.7], it follows that the open subannulus of A_n maps to a single residue disc. This holds for every n, and therefore we have that the open punctured disc maps to a single residue disc. The theorem follows. $\square$

8 Higher-dimensional extension theorem

In this section, we deduce the higher-dimensional extension theorem from the theorem for $\mathsf{D}^\times$.

8.1 Preliminaries

We first prove some general statements regarding mermorphic extension. We will work in the following setting. Let X be a quasi-projective algebraic variety over K with a projective compactification $\overline{X}$ over K. We fix a closed embedding $\overline{X} \hookrightarrow \mathbb{P}^n_K$ into projective n-space. This corresponds to the data of a very ample line bundle on $\overline{X}$ relative to K that we denote by $\mathscr{L}$, along with $n+1$ generating global sections $s_0, \dots, s_n \in H^0(\overline{X}, \mathscr{L})$.

Definition 8.1. We say that the compactification $X \subset \overline{X}$ *satisfies one-dimensional Borel extension over all finite extensions of* K, if for any finite field extension F of K, every analytic map $g^\times : \mathsf{D}^\times_F \to (X \times_K F)^{\mathrm{an}}$ over F admits an analytic extension $g : \mathsf{D}_F \to (\overline{X} \times_K F)^{\mathrm{an}}$.

Definition 8.2. Let W, Y be reduced rigid analytic spaces and $Z \subset Y$ a closed subspace. We say a morphism $f : Y \backslash Z \to W$ *extends meromorphically over* Z if the (metric) closure of the graph of f in $Y \times W$ is an analytic subspace. Equivalently, there is modification $g : Y' \to Y$ and a commutative diagram

$$\begin{tikzcd} & Y' \arrow[dl, "g"'] \arrow[dr] & \\ Y & & W \\ & Y\backslash Z \arrow[ul, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

where i is the inclusion.

The main result of this section is the following, which says that the one-dimensional Borel extension implies that analytic maps from poly-punctured disks $(\mathsf{D}^\times)^a \times \mathsf{D}^b$ into X extend meromoprhically.

Proposition 8.3. *Suppose $X \subset \overline{X}$ satisfies one-dimensional Borel extension over all finite extensions of K. Then, given any finite field extension F of K, and any analytic map $h^\times : (\mathsf{D}^\times_F)^a \times \mathsf{D}^b_F \to X^{\mathrm{an}}_F$, there is a closed analytic subspace $Z \subset \mathsf{D}^{a+b}_F$ of codimension at least 2 contained in the complement of $(\mathsf{D}^\times)^a \times \mathsf{D}^b$ such that $h^\times$ extends to an analytic map $h : \mathsf{D}^{a+b}_F \setminus Z \to \overline{X}^{\mathrm{an}}_F$. Moreover, h extends meromorphically over Z.*

We need two preparatory lemmas first. In Lemma 8.4, we show that there are no non-trivial line bundles on $(\mathsf{D}^\times)^a \times \mathsf{D}^b$, and in Lemma 8.5, we prove that an analytic function on $(\mathsf{D}^\times)^a \times \mathsf{D}^b$, whose restriction to every one-dimensional punctured disk extends meromorphically to the disk D, must itself extend meromorphically to D^{a+b}.

Lemma 8.4. *Let $\mathsf{A}_0 := \mathrm{Spa}(K\langle z^{\pm 1}\rangle, \mathcal{O}_K\langle z^{\pm 1}\rangle)$ denote the thin annulus over K at radius 1 centered at 0. Let a, b, c be non-negative integers. Then every line bundle on $(\mathsf{D}^\times)^a \times \mathsf{D}^b \times \mathsf{A}_0^c$ is trivial. In particular, every line bundle on $(\mathsf{D}^\times)^a \times \mathsf{D}^b$ is trivial.*

Proof. We induct on a with the base case of $a = 0$, being a consequence of the fact that the affinoid algebra $K\langle t_1, \ldots, t_b, z_1^{\pm 1}, \ldots, z_c^{\pm 1}\rangle$ is a UFD (see [vdP82, Theorem 3.25]). Suppose $a \geq 1$ and $\mathcal{E}$ is a line bundle on $(\mathsf{D}^\times)^a \times \mathsf{D}^b \times \mathsf{A}_0^c$. By the inductive hypothesis, the restriction of $\mathcal{E}$ to $(\mathsf{D}^\times)^{a-1} \times \mathsf{A}_0 \times \mathsf{D}^b \times \mathsf{A}_0^c$ is trivial. We may thus glue $\mathcal{E}$ with the trivial line bundle on $(\mathsf{D}^\times)^{a-1} \times (\mathbb{P}_K^1)^{\mathrm{an}} \setminus \{|z| < 1\} \times \mathsf{D}^b \times \mathsf{A}_0^c$ to obtain a line bundle $\hat{\mathcal{E}}$ on $(\mathsf{D}^\times)^{a-1} \times \mathbb{A}_K^{1,\mathrm{an}} \times \mathsf{D}^b \times \mathsf{A}_0^c$ that extends $\mathcal{E}$. By [Sig17, Prop. 3.6] (see also [KST20, Corollary 5] for the smooth affinoid case), $\hat{\mathcal{E}}$ is the pullback of a line bundle on $(\mathsf{D}^\times)^{a-1} \times \mathsf{D}^b \times \mathsf{A}_0^c$. However, by the inductive hypothesis every line bundle on $(\mathsf{D}^\times)^{a-1} \times \mathsf{D}^b \times \mathsf{A}_0^c$ is trivial and hence so is $\hat{\mathcal{E}}$ and therefore so is $\mathcal{E}$. $\square$

Lemma 8.5. *Let $f(\underline{z}, \underline{t})$ be an analytic function on $(\mathsf{D}^\times)^a \times \mathsf{D}^b$, such that for each $1 \leq i \leq a$, and each finite extension F of K, and every F-valued point $P' = (c_1, \ldots, \widehat{c_i}, \ldots, c_a, \underline{\tau}) \in (\mathsf{D}^\times)^{a-1} \times \mathsf{D}^b$, the specialization $f(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau}) \in \mathcal{O}(\mathsf{D}_F^\times)$ has only finitely many zeroes on $\mathsf{D}_F^\times$. Then $f(\underline{z}, \underline{t})$ extends to a meromorphic function on D^{a+b}.*

Proof. Fix $1 \leq i \leq a$. To simplify notations, we set $z' = (z_1, \ldots, \widehat{z_i}, \ldots, z_n)$. The function f admits a power series development

$$f(\underline{z}, \underline{t}) = \sum_{m \in \mathbb{Z}} a_m(z', \underline{t}) z_i^m,$$

for analytic functions $a_m(z', \underline{t}) \in \mathcal{O}((\mathsf{D}^\times)^{a-1} \times \mathsf{D}^b)$. By the p-adic big Picard theorem, since for each specialization P', the function $f(z_i, P')$ has only finitely many zeros in $\mathsf{D}_F^\times$, we have that for each such P', the function $f(z_i, P')$ cannot have an essential singularity at $z_i = 0$. That is to say that $a_m(P') = 0$ for all m sufficiently small. In other words, there is an $m_0 \in \mathbb{Z}$ such that $P' \in V(\{a_l(z', \underline{t}) : l \leq m_0\})$. This being true for every classical point P', implies that $(\mathsf{D}^\times)^{a-1} \times \mathsf{D}^b = \cup_{m \in \mathbb{Z}} V(\{a_l(z', \underline{t}) : l \leq m\})$. By the Baire category theorem, this implies that there is an $m(i) \in \mathbb{Z}$ such that for $l < m(i)$, $a_l(z', \underline{t}) = 0$. In particular, $z_i^{-m(i)} f(\underline{z}, \underline{t})$, extends analytically across the locus $z_i = 0$ as well. The function $\prod_{1 \leq j \leq a} z_j^{-m(j)} f(\underline{z}, \underline{t})$ thus extends analytically to the complement inside D^{a+b} of the codimension 2 subvariety $\cup_{1 \leq r < s \leq a} V(z_r, z_s)$. Hence by the non-archimedean Hartog extension theorem [Bar75], defines an analytic function on D^{a+b}. This completes the proof. $\square$

Proof of Proposition 8.3. We may assume that $F = K$. Let $h^\times : (\mathsf{D}^\times)^a \times \mathsf{D}^b \to X^{\mathrm{an}}$ be an analytic map. By Lemma 8.4, we may pick a trivialization $\phi : (h^\times)^*(\mathscr{L}^{\mathrm{an}}) \xrightarrow{\sim} \mathcal{O}_{(\mathsf{D}^\times)^a \times \mathsf{D}^b}$ of the pullback line bundle $(h^\times)^*(\mathscr{L}^{\mathrm{an}})$. We let $\underline{z} := (z_1, \ldots, z_a)$ denote the coordinates on $(\mathsf{D}^\times)^a$ and $\underline{t} := (t_1, \ldots, t_b)$ the coordinates on D^b. Let $f_r(\underline{z}, \underline{t}) := \phi(s_r) \in \mathcal{O}((\mathsf{D}^\times)^a \times \mathsf{D}^b)$. On classical points $(\underline{z}, \underline{t}) \in (\mathsf{D}^\times)^a \times \mathsf{D}^b$, the map $h^\times : (\mathsf{D}^\times)^a \times \mathsf{D}^b \to X^{\mathrm{an}} \hookrightarrow \mathbb{P}_K^{n,\mathrm{an}}$ is given in projective coordinates by $[f_0(\underline{z}, \underline{t}) : \ldots : f_n(\underline{z}, \underline{t})]$. For each $i \in \{1, \ldots, a\}$, and any finite extension F of K and an F-valued point $P' = (c_1, \ldots, \widehat{c_i}, \ldots, c_a, \underline{\tau}) \in (\mathsf{D}^\times)^{a-1} \times \mathsf{D}^b$, the analytic map $h_{P'}^\times : \mathsf{D}_F^\times \to \mathbb{P}_F^{n,\mathrm{an}}$ given by $z_i \mapsto [f_0(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau}) : \ldots : f_n(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau})]$, by hypothesis extends analytically to a map $h_{P'} : \mathsf{D}_F \to \mathbb{P}_F^{n,\mathrm{an}}$. In particular, there exist analytic functions $g_r(z_i) \in F\langle z_i\rangle$ (depending on P') with no common zeroes on D_F such that for all classical points $z_i \in \mathsf{D}_F^\times$, $[f_0(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau}) : \ldots : f_n(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau})] = [g_0(z_i) : \ldots : g_n(z_i)]$. This implies that for all r, s, $f_r(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau}) g_s(z_i) = f_s(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau}) g_r(z_i)$ as analytic functions on $\mathsf{D}_F^\times$ (with coordinate z_i). Since the $\{f_r(c_1, \ldots, z_i, \ldots, c_a, \underline{\tau})\}_{0 \leq r \leq n}$ and the

$\{g_r(z_i)\}_{0\le r\le n}$ have no common zeroes in $\mathsf{D}_F^\times$, this in particular implies that for each $0 \le r \le n$, the zero set $V(f_r(c_1,\dots,z_i,\dots,c_a,\underline{\tau}))$ equals $V(g_r(z_i))$ in $\mathsf{D}_F^\times$ and is thus *finite*. We conclude from Lemma 8.5 that there are non-negative integers $\{m(j) : 1 \le j \le a\}$, such that for each $0 \le r \le n$, $F_r(\underline{z},\underline{t}) := \prod_{1\le j\le a} z_j^{m(j)} f_r(\underline{z},\underline{t})$ extends analytically to D^{a+b} and such that for each $1 \le j \le a$, there is an r such that $z_j \nmid F_r(\underline{z},\underline{t})$ in $\mathcal{O}(\mathsf{D}^{a+b})$. The set of common zeroes $Z := V(\{F_r(\underline{z},\underline{t}) : 0 \le r \le n\})$ has codimension at least 2 in D^{a+b}, and $h^\times$ extends analytically to the map $h : \mathsf{D}^{a+b} \setminus Z \to \mathbb{P}_K^{n,\mathrm{an}}$, given by $(\underline{z},\underline{t}) \mapsto [F_0(\underline{z},\underline{t}) : \dots : F_n(\underline{z},\underline{t})]$.

For the final claim, let $V \subset \mathsf{D}^{a+b} \times \mathbb{P}_K^{n,\mathrm{an}}$ be the subspace cut out by $x_iF_j - x_jF_i$, where the x_i are homogeneous coordinates on $\mathbb{P}_K^n$. This subspace is equal to the graph Γ of h when intersected with $U := (\mathsf{D}^{a+b}\backslash Z) \times \mathbb{P}_K^{n,\mathrm{an}}$. There is therefore an irreducible component V_0 of V for which $\Gamma = U \cap V_0$ ([Con99, Cor. 2.2.9]), and since $U \cap V_0$ is metrically dense in V_0, it follows that V_0 is the closure of Γ. ☐

Remark 8.6. We remark that the meromorphicity in Proposition 8.3 is automatic for an analytic morphism defined outside codimension 2, as in the complex analytic case:

Lemma 8.7. *Let Y be a smooth connected rigid analytic space and $Z \subset Y$ a closed subspace of codimension ≥ 2. Then any morphism $f : Y\backslash Z \to X^{\mathrm{an}}$ extends meromorphically over Z.*

Proof. We may assume $X = \mathbb{P}_K^n$. Since Z has codimension at least 2 in Y, the pull-back $f^*(\mathcal{O}(1))$ extends to a line bundle on Y (using the correspondence between line bundles and Weil divisors and applying Remmert-Stein which allows us to extend Weil divisors outside the codimension $\ge$ 2 analytic subvariety Z.) The question of whether f extends meromorphically across Z to all of Y is local on Y. We may therefore assume that the pull-back $f^*\mathcal{O}(1)$ is trivial. Then the morphism $f : Y \setminus Z \to \mathbb{P}_K^{n,\mathrm{an}}$ is described in homogenous coordinates by $n+1$ analytic functions $y \mapsto [f_0(y) : \dots : f_n(y)]$ where $f_i(y) \in \mathcal{O}(Y \setminus Z)$ do not have any common zeroes in $Y \setminus Z$. By the non-archimedean Hartog extension theorem, $f_i(y)$ uniquely extend to analytic functions $F_i \in \mathcal{O}(Y)$. The subspace in $Y \times \mathbb{P}_K^{n,\mathrm{an}}$ cut out by $x_iF_j - x_jF_i$ contains as an irreducible component the closure inside $Y \times \mathbb{P}_K^{n,\mathrm{an}}$ of the graph of $f : Y \setminus Z \to \mathbb{P}_K^{n,\mathrm{an}}$. ☐

8.2 Higher-dimensional extension for Shimura varieties and period images

Let X denote either $\mathrm{Sh}_{\mathsf{K}}(G, \mathbf{X})$ or a geometric period image Y as described in §1. For the reader's convenience we summarize here the structures that will be relevant for the proof of Theorem 1.1.

- A $\mathbb{Z}_p$-local system $\mathbb{V}_{\text{ét},p}$ on X.
- A filtered vector bundle $V_{\mathrm{dR}} := (\mathcal{V}, F^\bullet)$ on X.
- A normal compactification X^{BB} of X for which the kth (for some k) power of the Griffiths bundle

$$\mathrm{Griff}(V_{\mathrm{dR}}) := \bigotimes_p \det F^p$$

  extends to an ample line bundle L. This is [BB66] in the Shimura variety case and [BFMT25, Thm 1.2] in the period image case.
- A log smooth proper $(X', D_{X'})$ with a modification $\pi_{X'\backslash D_{X'}} : X'\backslash D_{X'} \to X$, such that the pullback $U_{\mathrm{dR}} := (\mathcal{U}, \nabla, F^\bullet)$ of $(\mathcal{V}, F^\bullet)$ admits a flat connection with respect to which the filtration is Griffiths transverse and extends to a logarithmic flat vector bundle $\bar{U}_{\mathrm{dR}} :=$

$(\bar{\mathcal{U}}, \nabla, F^\bullet \bar{\mathcal{U}})$ such that the eigenvalues of the residues are contained in $[0,1) \cap \mathbb{Q}$. Moreover, $\pi_{X' \backslash D_{X'}} : X' \backslash D_{X'} \to X$ extends to a morphism $\pi : (X', D_{X'}) \to (X^{\mathrm{BB}}, X^{\mathrm{BB}} \backslash X)$ for which

$$\pi^* L \cong \mathrm{Griff}(\bar{U}_{\mathrm{dR}})^k.$$

Here, X' is taken to be the largest stratified resolution S^m in the period image case, and S itself in the Shimura variety case. Again, this follows from [BB66] in the Shimura variety case and [BFMT25, Thm 1.2] in the period image case.

- We have
$$\bar{U}^{\mathrm{an}}_{\mathrm{dR}} \cong D_{\mathrm{dR},\log}(\pi^*_{X' \backslash D_{X'}} \mathbb{V}_{\text{ét},p}) \tag{3}$$
via the p-adic Riemann–Hilbert correspondence of [DLLZ23, Thm 1.7]. In the period image case the isomorphism is given over P' by the second part of [DLLZ23, Thm 1.1], and in the Shimura variety case by [DLLZ23, Thm 1.5].

Proposition 8.8. *Let (M, D_M) be a log smooth rigid-analytic variety and $f : M \backslash D_M \to X^{\mathrm{an}}$ a morphism such that $M \backslash D_M \to X^{\mathrm{BB,an}}$ extends meromorphically over D_M. Then it extends regularly over D_M.*

Proof. By meromorphicity and embedded resolution of singularities [Tem18, Thms 1.1.9, 1.1.13], there is a log smooth $(M', D_{M'})$, a modification $g : (M', D_{M'}) \to (M, D_M)$, and a diagram

$$\begin{tikzcd}
 & (M', D_{M'}) \arrow[rr, "h"] \arrow[dd, "g"] & & (X', D_{X'}) \arrow[dd, "\pi"] \\
M' \backslash D_{M'} \arrow[ur] \arrow[dd, "g_{M' \backslash D_{M'}}"] & & & \\
 & (M, D_M) & & (X^{\mathrm{BB}}, X^{\mathrm{BB}} \backslash X) \\
M \backslash D_M \arrow[ur] \arrow[rr, "f"] & & X \arrow[ur] &
\end{tikzcd}$$

Since $(\pi h)^*_{M' \backslash D_{M'}} \mathbb{V} \cong (f g_{M' \backslash D_{M'}})^* \mathbb{V}_{\text{ét},p}$, we have

$$g^* D_{\mathrm{dR},\log}(f^* \mathbb{V}_{\text{ét},p}) \cong h^* D_{\mathrm{dR},\log}(\pi^*_{X' \backslash D_{X'}} \mathbb{V}_{\text{ét},p})$$

and likewise for the Griffiths bundles. Since $\mathrm{Griff}(D_{\mathrm{dR},\log}(\pi^*_{S' \backslash D_{S'}} \mathbb{V}_{\text{ét},p}))^k$ descends amply to X^{BB}, it follows that πh factors through g, by the rigidity lemma (see e.g. [Deb01, Lemma 1.15]). $\square$

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