LAGRANGIAN HYPERPLANES IN HOLOMORPHIC SYMPLECTIC VARIETIES: COMPUTATIONAL APPENDIX

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Abstract. We collect here the computations from [BJ11]. In that paper, we classify the cohomology classes of Lagrangian hyperplanes $\mathbb{P}^4$ in a smooth manifold X deformation equivalent to a Hilbert scheme of 4 points on a K3 surface, up to the monodromy action. Classically, the cone of effective curves on a K3 surface S is generated by nonegative classes C, for which $(C,C)\geq 0$, and nodal classes C, for which $(C,C)=-2$; Hassett and Tschinkel conjecture that the cone of effective curves on a holomorphic symplectic variety X is similarly controlled by "nodal" classes C such that $(C,C)=-\gamma$, for $(\cdot,\cdot)$ now the Beauville-Bogomolov form, where γ classifies the geometry of the extremal contraction associated to C. In particular, they conjecture that for X deformation equivalent to a Hilbert scheme of n points on a K3 surface, the class $C=\ell$ of a line in a smooth Lagrangian n-plane $\mathbb{P}^n$ must satisfy $(\ell,\ell)=-\frac{n+3}{2}$. We prove the conjecture for $n=4$ by computing the ring of monodromy invariants on X, and showing there is a unique monodromy orbit of Lagrangian hyperplanes.

1. Invariant Classes

(1.1) By [BJ11, §1.5],

$$\dim H^2(S^{[4]},\mathbb{Q})^{G_S}=1 \qquad \dim H^2(S^{[4]},\mathbb{Q})^{G_X}=0$$
$$\dim H^4(S^{[4]},\mathbb{Q})^{G_S}=4 \qquad \dim H^4(S^{[4]},\mathbb{Q})^{G_X}=2$$
$$\dim H^6(S^{[4]},\mathbb{Q})^{G_S}=5 \qquad \dim H^6(S^{[4]},\mathbb{Q})^{G_X}=1$$
$$\dim H^8(S^{[4]},\mathbb{Q})^{G_S}=8 \qquad \dim H^8(S^{[4]},\mathbb{Q})^{G_X}=3$$

(1.2) In the notation of [BJ11, §1.8], the invariant classes for $H^2(S^{[4]},\mathbb{Q})$ are:

$$\delta=I(\{1\}_2,\{1,1\}_1)=\sum_{(12)}1_{12}\otimes 1_3\otimes 1_4(12)$$

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(1.3) The invariant classes for $H^4(S^{[4]}, \mathbb{Q})$ are:

$$W = I(\{1\}_3, \{1\}_1) = \sum_{(123)} 1_{123} \otimes 1_4(123)$$
$$X = I(\{1,1\}_2) = \sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34)$$
$$Y = I(\{1,1,1,[\mathrm{pt}]\}_1) = \sum_{1} [\mathrm{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\mathrm{id})$$
$$Z = I(\{1,1,e,e^\vee\}_1) = \sum_{j,(12)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\mathrm{id})$$

(1.4) The invariant classes for $H^6(S^{[4]}, \mathbb{Q})$ are:

$$P = I(\{1\}_4) = \sum_{(1234)} 1_{1234}(1234)$$
$$Q = I(\{[\mathrm{pt}]\}_2, \{1,1\}_1) = \sum_{(12)} [\mathrm{pt}]_{12} \otimes 1_3 \otimes 1_4(12)$$
$$R = I(\{1\}_2, \{1,[\mathrm{pt}]\}_1) = \sum_{(12),3} 1_{12} \otimes [\mathrm{pt}]_3 \otimes 1_4(12)$$
$$S = I(\{e^\vee\}_2, \{e,1\}_1) = \sum_{j,1,(23)} (e_j)_1 \otimes (e_j^\vee)_{23} \otimes 1_4(23)$$
$$T = I(\{1\}_2, \{e,e^\vee\}_1) = \sum_{(12)} 1_{12} \otimes (e_j)_3 \otimes (e_j^\vee)_4(12)$$

(1.5) The invariant classes for $H^8(S^{[4]}, \mathbb{Q})$ are:

$$A = I(\{e\}_3, \{e^\vee\}_1) = \sum_{j,(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123)$$
$$B = I(\{1\}_3, \{[\mathrm{pt}]\}_1) = \sum_{(123)} 1_{123} \otimes [\mathrm{pt}]_4(123)$$
$$C = I(\{[\mathrm{pt}]\}_3, \{1\}_1) = \sum_{(123)} [\mathrm{pt}]_{123} \otimes 1_4(123)$$
$$D = I(\{1, [\mathrm{pt}]\}_2) = \sum_{(12)} [\mathrm{pt}]_{12} \otimes 1_{34}(12)(34)$$
$$E = I(\{e, e^\vee\}_2) = \sum_{j,(12)(34)} (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34)$$
$$F = I(\{1, 1, [\mathrm{pt}], [\mathrm{pt}]\}_1) = \sum_{(12)} [[\mathrm{pt}]]_1 \otimes [[\mathrm{pt}]]_2 \otimes 1_3 \otimes 1_4(\mathrm{id})$$
$$G = I(\{1, e, e^\vee, [\mathrm{pt}]\}_1) = \sum_{j,1,(23)} [[\mathrm{pt}]]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3(\mathrm{id})$$
$$H = I(\{e, e, e^\vee, e^\vee\}_1) = \sum_{j,k,(12)(34)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4 \cdot \mathrm{id}$$

2. The Ring $A\{S_4\}$

(2.1) Below, for various pairs of π and σ we give the relevant orbits and defect, *cf.* [BJ11, §1.1]:

π	σ	$\pi \cdot \sigma$	$\langle \pi \rangle \backslash [n]$	$\langle \sigma \rangle \backslash [n]$	$\langle \pi, \sigma \rangle \backslash [n]$	$\langle \pi \cdot \sigma \rangle \backslash [n]$	$g(\pi, \sigma)$
(id)	σ	σ	1, 2, 3, 4	*	*	*	1

π	σ	$\pi \cdot \sigma$	$\langle\pi\rangle\backslash[n]$	$\langle\sigma\rangle\backslash[n]$	$\langle\pi,\sigma\rangle\backslash[n]$	$\langle\pi\cdot\sigma\rangle\backslash[n]$	$g(\pi,\sigma)$
(12)	(12)	id	$(1,2),3,4$	$(1,2),3,4$	$(1,2),3,4$	$1,2,3,4$	$1_{1,2}\otimes 1_3\otimes 1_4$
(12)	(13)	(132)	$(1,2),3,4$	$(1,3),2,4$	$(1,2,3),4$	$(1,2,3),4$	$1_4\otimes 1_{1,2,3}$
(12)	(14)	(142)	$(1,2),3,4$	$(1,4),2,3$	$(1,2,4),3$	$(1,2,4),3$	$1_3\otimes 1_{1,2,4}$
(12)	(23)	(123)	$(1,2),3,4$	$1,4,(2,3)$	$(1,2,3),4$	$(1,2,3),4$	$1_4\otimes 1_{1,2,3}$
(12)	(24)	(124)	$(1,2),3,4$	$1,3,(2,4)$	$(1,2,4),3$	$(1,2,4),3$	$1_3\otimes 1_{1,2,4}$
(12)	(34)	(12)(34)	$(1,2),3,4$	$1,2,(3,4)$	$(1,2),(3,4)$	$(1,2),(3,4)$	$1_{1,2}\otimes 1_{3,4}$
(12)	(12)(34)	(34)	$(1,2),3,4$	$(1,2),(3,4)$	$(1,2),(3,4)$	$1,2,(3,4)$	$1_{1,2}\otimes 1_{3,4}$
(12)	(13)(24)	(1324)	$(1,2),3,4$	$(1,3),(2,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$1_{1,2,3,4}$
(12)	(14)(23)	(1423)	$(1,2),3,4$	$(1,4),(2,3)$	$(1,2,3,4)$	$(1,2,3,4)$	$1_{1,2,3,4}$
(12)	(123)	(23)	$(1,2),3,4$	$(1,2,3),4$	$(1,2,3),4$	$2,4,(1,3)$	$1_4\otimes 1_{1,2,3}$
(12)	(132)	(13)	$(1,2),3,4$	$(1,2,3),4$	$(1,2,3),4$	$1,(2,3),4$	$1_4\otimes 1_{1,2,3}$
(12)	(124)	(24)	$(1,2),3,4$	$(1,2,4),3$	$(1,2,4),3$	$(1,4),2,3$	$1_3\otimes 1_{1,2,4}$
(12)	(142)	(14)	$(1,2),3,4$	$(1,2,4),3$	$(1,2,4),3$	$1,3,(2,4)$	$1_3\otimes 1_{1,2,4}$
(12)	(134)	(1342)	$(1,2),3,4$	$(1,3,4),2$	$(1,2,3,4)$	$(1,2,3,4)$	$1_{1,2,3,4}$
(12)	(143)	(1432)	$(1,2),3,4$	$(1,3,4),2$	$(1,2,3,4)$	$(1,2,3,4)$	$1_{1,2,3,4}$
(12)	(234)	(1234)	$(1,2),3,4$	$1,(2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$1_{1,2,3,4}$
(12)	(243)	(1243)	$(1,2),3,4$	$1,(2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$1_{1,2,3,4}$
(12)	(1234)	(234)	$(1,2),3,4$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,3,4),2$	$1_{1,2,3,4}$
(12)	(1243)	(243)	$(1,2),3,4$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,3,4),2$	$1_{1,2,3,4}$
(12)	(1324)	(13)(24)	$(1,2),3,4$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,4),(2,3)$	$1_{1,2,3,4}$
(12)	(1342)	(134)	$(1,2),3,4$	$(1,2,3,4)$	$(1,2,3,4)$	$1,(2,3,4)$	$1_{1,2,3,4}$
(12)	(1423)	(14)(23)	$(1,2),3,4$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,3),(2,4)$	$1_{1,2,3,4}$
(12)	(1432)	(143)	$(1,2),3,4$	$(1,2,3,4)$	$(1,2,3,4)$	$1,(2,3,4)$	$1_{1,2,3,4}$

π	σ	$\pi \cdot \sigma$	$\langle\pi\rangle\backslash[n]$	$\langle\sigma\rangle\backslash[n]$	$\langle\pi,\sigma\rangle\backslash[n]$	$\langle\pi\cdot\sigma\rangle\backslash[n]$	$g(\pi,\sigma)$
(12)(34)	(12)(34)	id	$(1,2),(3,4)$	$(1,2),(3,4)$	$(1,2),(3,4)$	$1,2,3,4$	$1_{1,2}\otimes 1_{3,4}$
(12)(34)	(13)(24)	(14)(23)	$(1,2),(3,4)$	$(1,3),(2,4)$	$(1,2,3,4)$	$(1,4),(2,3)$	$1_{1,2,3,4}$
(12)(34)	(14)(23)	(13)(24)	$(1,2),(3,4)$	$(1,4),(2,3)$	$(1,2,3,4)$	$(1,3),(2,4)$	$1_{1,2,3,4}$
(12)(34)	(123)	(243)	$(1,2),(3,4)$	$4,(1,2,3)$	$(1,2,3,4)$	$2,(1,3,4)$	$1_{1,2,3,4}$
(12)(34)	(132)	(143)	$(1,2),(3,4)$	$4,(1,2,3)$	$(1,2,3,4)$	$1,(2,3,4)$	$1_{1,2,3,4}$
(12)(34)	(124)	(234)	$(1,2),(3,4)$	$3,(1,2,4)$	$(1,2,3,4)$	$2,(1,3,4)$	$1_{1,2,3,4}$
(12)(34)	(142)	(134)	$(1,2),(3,4)$	$3,(1,2,4)$	$(1,2,3,4)$	$1,(2,3,4)$	$1_{1,2,3,4}$
(12)(34)	(134)	(142)	$(1,2),(3,4)$	$2,(1,3,4)$	$(1,2,3,4)$	$4,(1,2,3)$	$1_{1,2,3,4}$
(12)(34)	(143)	(132)	$(1,2),(3,4)$	$2,(1,3,4)$	$(1,2,3,4)$	$3,(1,2,4)$	$1_{1,2,3,4}$
(12)(34)	(234)	(124)	$(1,2),(3,4)$	$1,(2,3,4)$	$(1,2,3,4)$	$4,(1,2,3)$	$1_{1,2,3,4}$
(12)(34)	(243)	(123)	$(1,2),(3,4)$	$1,(2,3,4)$	$(1,2,3,4)$	$3,(1,2,4)$	$1_{1,2,3,4}$
(12)(34)	(1234)	(24)	$(1,2),(3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$2,4,(1,3)$	$1_{1,2,3,4}$
(12)(34)	(1243)	(23)	$(1,2),(3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$2,3,(1,4)$	$1_{1,2,3,4}$
(12)(34)	(1324)	(1423)	$(1,2),(3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(12)(34)	(1342)	(14)	$(1,2),(3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$1,4,(2,3)$	$1_{1,2,3,4}$
(12)(34)	(1423)	(1324)	$(1,2),(3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(12)(34)	(1432)	(13)	$(1,2),(3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$1,3,(2,4)$	$1_{1,2,3,4}$

π	σ	$\pi\cdot\sigma$	$\langle\pi\rangle\backslash[n]$	$\langle\sigma\rangle\backslash[n]$	$\langle\pi,\sigma\rangle\backslash[n]$	$\langle\pi\cdot\sigma\rangle\backslash[n]$	$g(\pi,\sigma)$
(123)	(123)	(132)	$4,(1,2,3)$	$4,(1,2,3)$	$4,(1,2,3)$	$4,(1,2,3)$	$-24[\mathrm{pt}]_{1,2,3}\otimes 1_4$
(123)	(132)	id	$4,(1,2,3)$	$4,(1,2,3)$	$4,(1,2,3)$	$1,2,3,4$	$1_4\otimes 1_{1,2,3}$
(123)	(124)	(13)(24)	$4,(1,2,3)$	$3,(1,2,4)$	$(1,2,3,4)$	$(1,4),(2,3)$	$1_{1,2,3,4}$
(123)	(142)	(143)	$4,(1,2,3)$	$3,(1,2,4)$	$(1,2,3,4)$	$1,(2,3,4)$	$1_{1,2,3,4}$
(123)	(134)	(234)	$4,(1,2,3)$	$2,(1,3,4)$	$(1,2,3,4)$	$3,(1,2,4)$	$1_{1,2,3,4}$
(123)	(143)	(14)(23)	$4,(1,2,3)$	$2,(1,3,4)$	$(1,2,3,4)$	$(1,2),(3,4)$	$1_{1,2,3,4}$
(123)	(234)	(12)(34)	$4,(1,2,3)$	$1,(2,3,4)$	$(1,2,3,4)$	$(1,3),(2,4)$	$1_{1,2,3,4}$
(123)	(243)	(124)	$4,(1,2,3)$	$1,(2,3,4)$	$(1,2,3,4)$	$2,(1,3,4)$	$1_{1,2,3,4}$
(123)	(1234)	(1342)	$4,(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(123)	(1243)	(1324)	$4,(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(123)	(1324)	(24)	$4,(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4)$	$2,3,(1,4)$	$1_{1,2,3,4}$
(123)	(1342)	(34)	$4,(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4)$	$1,3,(2,4)$	$1_{1,2,3,4}$
(123)	(1423)	(1432)	$4,(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(123)	(1432)	(14)	$4,(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4)$	$1,2,(3,4)$	$1_{1,2,3,4}$

π	σ	$\pi\cdot\sigma$	$\langle\pi\rangle\backslash[n]$	$\langle\sigma\rangle\backslash[n]$	$\langle\pi,\sigma\rangle\backslash[n]$	$\langle\pi\cdot\sigma\rangle\backslash[n]$	$g(\pi,\sigma)$
(1234)	(1234)	(13)(24)	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$(1,3),(2,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(1234)	(1243)	(132)	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$3,(1,2,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(1234)	(1324)	(142)	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$2,(1,3,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(1234)	(1342)	(143)	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$1,(2,3,4)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(1234)	(1423)	(243)	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$4,(1,2,3)$	$-24[\mathrm{pt}]_{1,2,3,4}$
(1234)	(1432)	id	$(1,2,3,4)$	$(1,2,3,4)$	$(1,2,3,4)$	$1,2,3,4$	$1_{1,2,3,4}$

(2.2) Deducing from the previous orbit computations the multiplication structure in the ring $A\{S_4\}$ is easy. For example, we have the following:

$$\begin{aligned}
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{1,2,3}\otimes\beta'_4\cdot(123)\big) &= (\mathbf{e}\alpha\alpha')_{1,2,3}\otimes(\beta\beta')_4\cdot(132)\\
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{1,2,3}\otimes\beta'_4\cdot(132)\big) &= \Delta(\alpha\alpha')_{1,2,3}\otimes(\beta\beta')_4\cdot\mathrm{id}\\
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{1,2,4}\otimes\beta'_3\cdot(124)\big) &= \Delta(\alpha\beta\alpha'\beta')_{13,24}\cdot(13)(24)\\
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{1,2,4}\otimes\beta'_3\cdot(142)\big) &= \Delta(\alpha\beta\alpha'\beta')_{2,134}\cdot(143)\\
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{1,3,4}\otimes\beta'_2\cdot(134)\big) &= \Delta(\alpha\beta\alpha'\beta')_{1,234}\cdot(234)\\
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{1,3,4}\otimes\beta'_2\cdot(143)\big) &= \Delta(\alpha\beta\alpha'\beta')_{14,23}\cdot(14)(23)\\
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{2,3,4}\otimes\beta'_1\cdot(234)\big) &= \Delta(\alpha\beta\alpha'\beta')_{12,34}\cdot(12)(34)\\
(\alpha_{1,2,3}\otimes\beta_4\cdot(123))\cdot\big(\alpha'_{2,3,4}\otimes\beta'_1\cdot(243)\big) &= \Delta(\alpha\beta\alpha'\beta')_{3,124}\cdot(124)
\end{aligned}$$

where $\mathbf{e}=-24[\mathrm{pt}]$.

Also

$$\begin{aligned}
(12)(12) &= \Delta(1)_{1,2}\otimes 1_3\otimes 1_4(\mathrm{id})\\
(12)(13) &= 1_{123}\otimes 1_4(132)\\
(12)(23) &= 1_{123}\otimes 1_4(123)
\end{aligned}$$

(2.3) Let $e_j^\vee = \sum_k t_{jk} e_k$ and $e_j = \sum r_{jk} e_k^\vee$ such that the matrices (t_{jk}) and (r_{jk}) are inverses to each other. Thus

$$\sum_i t_{ij} r_{ik} = \delta_{j=k}$$

Also $(e_j, e_k) = (\sum r_{ji} e_i^\vee, e_k) = r_{jk}$ and $(e_j^\vee, e_k^\vee) = t_{jk}$. Moreover, $e_j e_k = (e_j, e_k) = r_{jk}[\mathrm{pt}]$ and $e_j^\vee e_k^\vee = t_{jk}[\mathrm{pt}]$.

3. Multiplication Table for Invariant Classes

(3.1) For computations using invariant classes one needs to construct a multiplication table which, because of symmetry, has $18 + \binom{18}{2} = 171$ entries. All formulas were either checked by SAGE code, available on either author's webpage, or by hand; many were checked by both.

3.2. δ. **Summary**:

$\delta^2 = 2X - 3Y - Z + 3W$	formula & sage
$\delta W = 4P - 4Q - 2R - 2S$	formula & sage
$\delta X = 2P - R - T$	sage
$\delta Y = 2Q + R$	sage
$\delta Z = 22Q + 2S + T$	sage
$\delta P = -3A - 3B - 3C - 4D - 4E$	formula & sage
$\delta Q = 3C + D - F$	sage
$\delta R = 3B + 3C + 2D - 4F - G$	sage
$\delta S = 3A + 66C + 4E - 2G$	sage
$\delta T = 3A + 22D - G - 2H$	formula

(3.2.1)

$$\begin{aligned}
\delta^2 &= \left(\sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4 (12) \right)^2 \\
&= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \left((12) + (13) + (14) + (23) + (24) + (34) \right) \\
&= \sum_{(12)} (\Delta(1)_{1,2} \otimes 1_3 \otimes 1_4(\mathrm{id}) + 1_{1,2,3} \otimes 1_4(132) \\
&+ 1_{1,2,4} \otimes 1_3(142) + 1_{1,2,3} \otimes 1_4(123) + 1_{1,2,4} \otimes 1_3(124) + 1_{12} \otimes 1_{34}(12)(34)) \\
&= -3 \sum_1 [\mathrm{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) - \sum_{(12)} \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) \\
&+ 3 \sum_{(123)} 1_{123} \otimes 1_4(123) + 2 \sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34) \\
&= -3Y - Z + 3W + 2X
\end{aligned}$$

(3.2.2)

$$\begin{aligned}
\delta W &= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \sum 1_{abc} \otimes 1_d(abc) \\
&= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \,(1_{123} \otimes 1_4(123) + 1_{124} \otimes 1_3(124) + 1_{134} \otimes 1_2(134) + 1_{234} \otimes 1_1(234) \\
&+ 1_{123} \otimes 1_4(132) + 1_{124} \otimes 1_3(142) + 1_{134} \otimes 1_2(143) + 1_{234} \otimes 1_1(243)) \\
&= \sum_{(12)} (\Delta(1)_{1,23} \otimes 1_4(23) + \Delta(1)_{1,24} \otimes 1_3(24) + 1_{1234}(1342) + 1_{1234}(1234) \\
&+ \Delta(1)_{2,13} \otimes 1_4(13) + \Delta(1)_{2,14} \otimes 1_3(1,4) + 1_{1234}(1432) + 1_{1234}(1243)) \\
&= 2 \sum_{1,(23)} \Delta(1)_{1,23}(23) + 4 \sum_{(1234)} 1_{1234}(1234) \\
&= 4 \sum_{(12)} 1_{1234}(1234) - 2 \sum_{1,(23)} 1_1 \otimes [\mathrm{pt}]_{23}(23) - 2 \sum_{1,(23)} [\mathrm{pt}]_1 \otimes 1_{23}(23) - 2 \sum_{1,(23)} \sum_{j=1}^{22} (e_j)_1 \otimes (e_j^\vee)_{23}(23) \\
&= 4P - 4Q - 2R - 2S
\end{aligned}$$

(3.2.3)

$$\begin{aligned}
\delta X &= \sum_{(12)} (12) \sum_{(12)(34)} (12)(34) \\
&= \sum_{(12)} (\Delta(1)_{1,2} \otimes 1_{34}(34) + (1324) + (1423)) \\
&= \sum_{(12)} \left(-[\mathrm{pt}]_1 \otimes 1_2 \otimes 1_{34}(34) - 1_1 \otimes [\mathrm{pt}]_2 \otimes 1_{34}(34) - \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_{34}(34) + (1324) + (1423) \right) \\
&= -R - T + 2P
\end{aligned}$$

(3.2.4)

$$\begin{aligned}
\delta Y &= \sum_{(12)} (12) \,([\mathrm{pt}]_1 + [\mathrm{pt}]_2 + [\mathrm{pt}]_3 + [\mathrm{pt}]_4)\,(\mathrm{id}) \\
&= \sum_{(12)} (2[\mathrm{pt}]_{12} \otimes 1_3 \otimes 1_4(12) + 1_{12} \otimes [\mathrm{pt}]_3 \otimes 1_4(12) + 1_{12} \otimes 1_3 \otimes [\mathrm{pt}]_4(12)) \\
&= 2Q + R
\end{aligned}$$

(3.2.5)

$$\delta Z = \sum_{(12)} \sum_{j} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\mathrm{id})\left((12)+(13)+(14)+(23)+(24)+(34)\right)$$
$$= \sum_{(12)} \sum_{j} \big((e_j e_j^\vee)_{12} \otimes 1_3 \otimes 1_4(\mathrm{id}) + (e_j)_{13} \otimes (e_j^\vee)_2 \otimes 1_4(13) + (e_j)_{14} \otimes (e_j^\vee)_2 \otimes 1_3(14)$$
$$+ (e_j)_1 \otimes (e_j^\vee)_{23} \otimes 1_4(23) + (e_j)_1 \otimes (e_j^\vee)_{24} \otimes 1_3(24) + (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_{34}(34)\big)$$
$$= 22Q + 2S + T$$

(3.2.6)

$$\delta P = \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12) \sum 1_{1234}(abcd)$$
$$= \sum_{(12)} 1_{12} \otimes 1_3 \otimes 1_4(12)\left((1234)+(1243)+(1342)+(1324)+(1432)+(1423)\right)$$
$$= \sum_{(12)} \left((12)(1234)+(12)(1243)+(12)(1342)+(12)(1324)+(12)(1432)+(12)(1423)\right)$$
$$= \sum_{(12)} \left(\Delta(1)_{1,234}(234)+\Delta(1)_{1,234}(243)+\Delta(1)_{2,134}(134)+\Delta(1)_{13,24}(13)(24)+\Delta(1)_{2,134}(143)+\Delta(1)_{14,23}\right.$$
$$= 3\sum_{(234)} \Delta(1)_{1,234}(234) + 4 \sum_{(13)(24)} \Delta(1)_{13,24}(13)(24)$$
$$= -3\sum_{(234)} 1_1 \otimes [\mathrm{pt}]_{234}(234) - 3\sum_{(234)} [\mathrm{pt}]_1 \otimes 1_{234}(234) - 3\sum_{(234)} \sum_{j=1}^{22} (e_j)_1 \otimes (e_j^\vee)_{234}(234)$$
$$-4\sum_{(13)(24)} 1_{13} \otimes [\mathrm{pt}]_{24}(13)(24) - 4\sum_{(13)(24)} [\mathrm{pt}]_{13} \otimes 1_{24}(13)(24) - 4\sum_{(13)(24)} \sum_{j=1}^{22} (e_j)_{13} \otimes (e_j^\vee)_{24}(13)(24)$$
$$= -3A - 3B - 3C - 4D - 4E$$

(3.2.7)

$$\delta Q = \sum_{(12)} [\mathrm{pt}]_{12} \otimes 1_3 \otimes 1_4(12)((12)+(13)+(14)+(23)+(24)+(34))$$
$$= \sum_{(12)} \left(\Delta([\mathrm{pt}])_{1,2} \otimes 1_3 \otimes 1_4(\mathrm{id}) + [\mathrm{pt}]_{123} \otimes 1_4(132) + [\mathrm{pt}]_{124} \otimes 1_3(142) + [\mathrm{pt}]_{123} \otimes 1_4(123) + [\mathrm{pt}]_{124} \otimes 1_3(\right.$$
$$\left. + [\mathrm{pt}]_{12} \otimes 1_{34}(12)(34)\right)$$
$$= -\sum_{(12)} [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) + 3\sum_{(123)} [\mathrm{pt}]_{123} \otimes 1_4(123) + \sum_{(12)} [\mathrm{pt}]_{12} \otimes 1_{34}(12)(34)$$
$$= -F + 3C + D$$

(3.2.8)

$$
\begin{aligned}
\delta R &= \sum_{(12),3} 1_{12} \otimes [\mathrm{pt}]_3 \otimes 1_4 (12)((12)+(13)+(14)+(23)+(24)+(34)) \\
&= \sum_{(12),3} (\Delta(1)_{1,2} \otimes [\mathrm{pt}]_3 \otimes 1_4(\mathrm{id}) + [\mathrm{pt}]_{123} \otimes 1_4(132) + 1_{124} \otimes [\mathrm{pt}]_3(142) + [\mathrm{pt}]_{123} \otimes 1_4(123) \\
&+ 1_{124} \otimes [\mathrm{pt}]_3(124) + 1_{12} \otimes [\mathrm{pt}]_{34}(12)(34)) \\
&= -4\sum_{(12)} [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) - \sum_{(12),3} \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \otimes [\mathrm{pt}]_3 \otimes 1_3(\mathrm{id}) \\
&+ 3\sum_{(123)} [\mathrm{pt}]_{123} \otimes 1_4(123) + 3\sum_{(123)} 1_{123} \otimes [\mathrm{pt}]_4(123) + 2\sum_{(12)} [\mathrm{pt}]_{12} \otimes 1_{34}(12)(34) \\
&= -4F - G + 3B + 3C + 2D
\end{aligned}
$$

(3.2.9)

$$
\begin{aligned}
\delta S &= \sum_{(12),3} \sum_j (e_j)_{12} \otimes (e_j^\vee)_3 \otimes 1_4(12)((12)+(13)+(14)+(23)+(24)+(34)) \\
&= \sum_{(12),3} \sum_j (\Delta(e_j)_{1,2} \otimes (e_j^\vee)_3 \otimes 1_4(\mathrm{id}) + (e_j e_j^\vee)_{123} \otimes 1_4(132) + (e_j)_{124} \otimes (e_j^\vee)_3(142) + (e_j e_j^\vee)_{123} \otimes 1_4(123) \\
&+ (e_j)_{124} \otimes (e_j^\vee)_3(124) + (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34)) \\
&= -\sum_{(12),3} \sum_j ((e_j)_1 \otimes [\mathrm{pt}]_2 \otimes (e_j^\vee)_3 \otimes 1_4 + [\mathrm{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4)(\mathrm{id}) + 3\sum_j \sum_{(123)} [\mathrm{pt}]_{123} \otimes 1_4(123) \\
&+ 3\sum_{(123)} \sum_j (e_j)_{123} \otimes (e_j^\vee)_4(123) + 4\sum_{(12)(34)} \sum_j (e_j)_{12}(e_j^\vee)_{34}(12)(34) \\
&= -2G + 66C + 3A + 4E
\end{aligned}
$$

(3.2.10)

$$
\begin{aligned}
\delta T &= \sum_{j,(12)} 1_{12} \otimes (e_j)_3 \otimes (e_j^\vee)_4(12)\left((12)+(13)+(14)+(23)+(24)+(34)\right) \\
&= \sum_{j,(12)} \big(\Delta(1)_{1,2} \otimes (e_j)_3 \otimes (e_j^\vee)_3(\mathrm{id}) + (e_j)_{123} \otimes (e_j^\vee)_4(132) + (e_j)_3 \otimes (e_j^\vee)_{124}(142) + (e_j)_{123} \otimes (e_j^\vee)_4(123) \\
&\quad + (e_j)_3 \otimes (e_j^\vee)_{124}(124) + 1_{12} \otimes [\mathrm{pt}]_{34}(12)(34)\big) \\
&= \sum_{j,(34)} \left(-[\mathrm{pt}]_1 \otimes 1_2 - 1_1 \otimes [\mathrm{pt}]_2 - \sum_k (e_k)_1 \otimes (e_k^\vee)_2\right) \otimes (e_j)_3 \otimes (e_j^\vee)_3(\mathrm{id}) + \sum_j \sum_{(12)} 1_{12} \otimes [\mathrm{pt}]_{34}(12)(34) \\
&\quad + 3 \sum_{j,(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) \\
&= -\sum_{j,1,(34)} [\mathrm{pt}]_1 \otimes 1_2 \otimes (e_j)_3 \otimes (e_j^\vee)_3(\mathrm{id}) - 2 \sum_{j,k,(12)(34)} (e_k)_1 \otimes (e_k^\vee)_2 \otimes (e_j)_3 \otimes (e_j^\vee)_3(\mathrm{id}) \\
&\quad + 22 \sum_{(12)} 1_{12} \otimes [\mathrm{pt}]_{34}(12)(34) + 3 \sum_{j,(123)} (e_j)_{123} \otimes (e_j^\vee)_4(123) \\
&= -G - 2H + 22D + 3A
\end{aligned}
$$

3.3. W. **Summary**:

$$
\begin{aligned}
W^2 &= -3A - 3B - 27C - 8D - 8E + 4F + 2G && \text{formula \& sage} \\
WX &= -3A - 3B - 3C && \text{sage} \\
WY &= B + 3C && \text{sage} \\
WZ &= 3A + 66C && \text{sage}
\end{aligned}
$$

(3.3.1)

$$
\begin{aligned}
W^2 &= \left(\sum_{(123)} 1_{123} \otimes 1_4(123) \right)^2 \\
&= \sum_{(123)} (123)\left((123)+(132)+(124)+(142)+(134)+(143)+(234)+(243)\right) \\
&= \sum_{(123)} (-24[\mathrm{pt}]_{123} \otimes 1_4(132) + \Delta(1)_{1,2,3} \otimes 1_4(\mathrm{id}) + \Delta(1)_{13,24}(13)(24) + \Delta(1)_{2,134}(143) + \Delta(1)_{1,234}(234) \\
&+ \Delta(1)_{14,23}(14)(23) + \Delta(1)_{12,34}(12)(34) + \Delta(1)_{3,124}(124)) \\
&= \sum_{(123)} (-24[\mathrm{pt}]_{123} \otimes 1_4(132) + \Delta(1)_{1,2,3} \otimes 1_4(\mathrm{id})) + 8 \sum_{(12)(34)} \Delta(1)_{12,34}(12)(34) + 3 \sum_{(234)} \Delta(1)_{1,234}(234) \\
&= -24 \sum_{(123)} ([\mathrm{pt}]_{123} \otimes 1_4(132)) + \sum_{(123)} \sum_{r=2}^{4} ([\mathrm{pt}]_1 \otimes [\mathrm{pt}]_r \otimes 1 \otimes 1 + \sum_j [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_r \otimes e_j \otimes e_j^\vee)(\mathrm{id}) \\
&- 8 \sum_{(12)(34)} ([\mathrm{pt}]_{12} \otimes 1_{34} + 1_{12} \otimes [\mathrm{pt}]_{34})(12)(34) - 8 \sum_{(12)(34)} \sum_j (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \\
&- 3 \sum_{(234)} [\mathrm{pt}]_{234} \otimes 1_1(234) - 3 \sum_{(234)} 1_{234} \otimes [\mathrm{pt}]_1(234) - 3 \sum_{(234)} \sum_j (e_j)_{234} \otimes (e_j^\vee)_1(234) \\
&= -3A - 3B - 27C - 8D - 8E + 4F + 2G
\end{aligned}
$$

3.4. X. **Summary**:

$X^2 = -2D - 2E + 2F + G + H$	formula & sage
$XY = 2D$	sage
$XZ = 22D + 4E$	sage

(3.4.1)

$$
\begin{aligned}
X^2 &= \left(\sum_{(12)(34)} 1_{12} \otimes 1_{34}(12)(34) \right)^2 \\
&= \sum_{(12)(34)} (\Delta(1)_{1,2} \otimes \Delta(1)_{3,4}(\mathrm{id}) + \Delta(1)_{14,23}(14)(23) + \Delta(1)_{13,24}(13)(24)) \\
&= \sum_{(12)(34)} \Delta(1)_{1,2} \otimes \Delta(1)_{3,4}(\mathrm{id}) + 2 \sum_{(12)(34)} \Delta(1)_{12,34}(12)(34) \\
&= -2 \sum_{(12)(34)} ([\mathrm{pt}]_{12} \otimes 1_{34} + 1_{12} \otimes [\mathrm{pt}]_{34})(12)(34) - 2 \sum_{(12)(34)} \sum_j (e_j)_{12} \otimes (e_j^\vee)_{34}(12)(34) \\
&+ \sum_{(12)(34)} \left(-[\mathrm{pt}]_1 \otimes 1_2 - 1_1 \otimes [\mathrm{pt}]_2 - \sum_j (e_j)_1 \otimes (e_j^\vee)_2 \right) \otimes \\
&\otimes \left(-[\mathrm{pt}]_3 \otimes 1_4 - 1_3 \otimes [\mathrm{pt}]_4 - \sum_k (e_k)_3 \otimes (e_k^\vee)_4 \right) (\mathrm{id}) \\
&= -2D - 2E + 2 \sum_{(12)} [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) + \sum_{(12)(34)} \sum_{j,k} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4(\mathrm{id}) \\
&+ \sum_{1,(23)} \sum_j [\mathrm{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4 \\
&= -2D - 2E + 2F + G + H
\end{aligned}
$$

3.5. Y. **Summary**:

$Y^2 = 2F$	formula & sage
$YZ = G$	sage

(3.5.1)

$$
\begin{aligned}
Y^2 &= \left(\sum_1 [\mathrm{pt}]_1 \otimes 1_2 \otimes 1_3 \otimes 1_4(\mathrm{id}) \right)^2 \\
&= \sum_1 ([\mathrm{pt}] \otimes [\mathrm{pt}] \otimes 1 \otimes 1 + [\mathrm{pt}] \otimes 1 \otimes [\mathrm{pt}] \otimes 1 + [\mathrm{pt}] \otimes 1 \otimes 1 \otimes [\mathrm{pt}]) \\
&= 2 \sum_{(12)} [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes 1 \otimes 1 \\
&= 2F
\end{aligned}
$$

3.6. Z. **Summary**:

$Z^2 = 22F + 2G + 2H$	formula

(3.6.1)

$$
\begin{aligned}
Z^2 &= \left(\sum_j \sum_{(12)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\mathrm{id})\right)^2 \\
&= \sum_{j,k} \sum_{(12)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes 1_3 \otimes 1_4(\mathrm{id})\, (e_k \otimes e_k^\vee \otimes 1 \otimes 1 + e_k \otimes 1 \otimes e_k^\vee \otimes 1 + e_k \otimes 1 \otimes 1 \otimes e_k^\vee \\
&+ 1 \otimes e_k \otimes e_k^\vee \otimes 1 + 1 \otimes e_k \otimes 1 \otimes e_k^\vee + 1 \otimes 1 \otimes e_k \otimes e_k^\vee) \\
&= \sum_{j,k,(12)} \big((e_j e_k) \otimes (e_j^\vee e_k^\vee) \otimes 1 \otimes 1 + (e_j e_k) \otimes e_j^\vee \otimes e_k^\vee \otimes 1 + (e_j e_k) \otimes e_j^\vee \otimes 1 \otimes e_k^\vee \\
&+ e_j \otimes e_j^\vee e_k \otimes e_k^\vee \otimes 1 + e_j \otimes e_j^\vee e_k \otimes 1 \otimes e_k^\vee + e_j \otimes e_j^\vee \otimes e_k \otimes e_k^\vee\big) \\
&= \sum_{j,k,(12)} t_{jk} r_{jk} [\mathrm{pt}] \otimes [\mathrm{pt}] \otimes 1 \otimes 1 + \sum_{j,k,(12),3} r_{jk} [\mathrm{pt}] \otimes (\sum_u t_{ju} e_u) \otimes e_k^\vee \otimes 1 \\
&+ \sum_{j,(12),3} e_j \otimes [\mathrm{pt}] \otimes e_k^\vee \otimes 1 + \sum_{j,k,(12)} e_j \otimes e_j^\vee \otimes e_k \otimes e_k^\vee \\
&= 22F + \sum_{k,u,(12),3} [\mathrm{pt}] \otimes e_u \otimes e_k^\vee \otimes 1 \sum_j r_{jk} t_{ju} + G + 2H \\
&= 22F + \sum_{k,u,(12),3} [\mathrm{pt}] \otimes e_u \otimes e_k^\vee \otimes 1 \delta_{u=k} + G + 2H \\
&= 22F + \sum_{k,(12),3} [\mathrm{pt}] \otimes e_k \otimes e_k^\vee \otimes 1 + G + 2H \\
&= 22F + 2G + 2H
\end{aligned}
$$

since $e_j e_k^\vee = \delta_{j=k}[\mathrm{pt}]$.

3.7. *A*. **Summary**:

$A^2 = 8 \cdot 22 [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id})$	formula
$AB = 0$	formula & sage
$AC = 0$	formula & sage
$AD = 0$	formula & sage
$AE = 0$	formula & sage
$AF = 0$	formula
$AG = 0$	formula
$AH = 0$	formula

(3.7.1)

$$\begin{aligned}
A^2 &= (\sum_{j=1}^{22}\sum_{(123)}(e_j)_{123}\otimes(e_j^\vee)_4(123))^2\\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(123)}(e_j)_{123}\otimes(e_j^\vee)_4(123)\,((e_k)_{123}\otimes(e_k^\vee)_4(123)+(e_k)_{123}\otimes(e_k^\vee)_4(132)\\
&+(e_k)_{124}\otimes(e_k^\vee)_3(124)+(e_k)_{124}\otimes(e_k^\vee)_3(142)+(e_k)_{134}\otimes(e_k^\vee)_2(134)\\
&+(e_k)_{134}\otimes(e_k^\vee)_2(143)+(e_k)_{234}\otimes(e_k^\vee)_1(234)+(e_k)_{234}\otimes(e_k^\vee)_1(243))\\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(123)}(e_j)_{123}\otimes(e_j^\vee)_4(123)(e_k)_{123}\otimes(e_k^\vee)_4(132)\\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(123)}\Delta(e_je_k)_{1,2,3}\otimes(e_j^\vee e_k^\vee)_4(\mathrm{id})\\
&= \sum_{(123)}\sum_{j=1}^{22}\sum_{k=1}^{22}(e_j,e_k)(e_j^\vee,e_k^\vee)[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id})\\
&= \sum_{(123)}22[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id})\\
&= 8\cdot 22[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id})
\end{aligned}$$

as $g((123),(123)) = -24[\mathrm{pt}]_{123}\otimes 1_4$ while all the other defects are trivial.

(3.7.2) All other products are zero as the only terms with nonzero product in middle cohomology are of the form

$$A^\sigma\cdot\sigma\otimes A^{\sigma^{-1}}\cdot\sigma^{-1}\to A^{\mathrm{id}}\cdot\mathrm{id}$$

Similarly below.

3.8. B**. Summary**:

$B^2 = 0$	formula
$BC = 8[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id})$	sage
$BD = 0$	formula & sage
$BE = 0$	formula & sage
$BF = 0$	formula & sage
$BG = 0$	formula & sage
$BH = 0$	formula

(3.8.1)

$$\begin{aligned}B^2 &= (\sum_{(123)} 1_{123} \otimes [\mathrm{pt}]_4(123))^2 \\ &= \sum_{(123)} 1_{123} \otimes [\mathrm{pt}]_4(123)\,(1_{123} \otimes [\mathrm{pt}]_4(123) + 1_{123} \otimes [\mathrm{pt}]_4(132) + 1_{124} \otimes [\mathrm{pt}]_3(124) \\ &+ 1_{124} \otimes [\mathrm{pt}]_3(142) + 1_{134} \otimes [\mathrm{pt}]_2(134) + 1_{134} \otimes [\mathrm{pt}]_2(143) + 1_{234} \otimes [\mathrm{pt}]_1(234) + 1_{234} \otimes [\mathrm{pt}]_1(243)) \\ &= 0\end{aligned}$$

3.9. C. **Summary**:

$C^2 = 0$	formula & sage
$CD = 0$	formula & sage
$CE = 0$	formula & sage
$CF = 0$	formula & sage
$CG = 0$	formula & sage
$CH = 0$	formula

3.10. D. **Summary**:

$D^2 = 6[\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id})$	sage
$DE = 0$	formula
$DF = 0$	formula
$DG = 0$	formula
$DH = 0$	formula

3.11. E. **Summary**:

$E^2 = 66[\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id})$	formula & sage
$EF = 0$	formula
$EG = 0$	formula
$EH = 0$	formula

(3.11.1)

$$
\begin{aligned}
E^2 &= (\sum_{j=1}^{22}\sum_{(12)(34)}(e_j)_{12}\otimes(e_j^\vee)_{34}(12)(34))^2\\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(12)(34)}(e_j)_{12}\otimes(e_j^\vee)_{34}(12)(34)\times\\
&\times((e_k)_{12}\otimes(e_k^\vee)_{34}(12)(34)+(e_k)_{13}\otimes(e_k^\vee)_{24}(13)(24)+(e_k)_{14}\otimes(e_k^\vee)_{23}(14)(23))\\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(12)(34)}(e_j)_{12}\otimes(e_j^\vee)_{34}(12)(34)(e_k)_{12}\otimes(e_k^\vee)_{34}(12)(34)\\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(12)(34)}\Delta(e_je_k)_{1,2}\otimes\Delta(e_j^\vee e_k^\vee)_{3,4}(\mathrm{id})\\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(12)(34)}r_{jk}t_{jk}[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id})\\
&= 3\sum_{j=1}^{22}\sum_{k=1}^{22}t_{jk}r_{jk}[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id})\\
&= 66[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id})
\end{aligned}
$$

3.12. F. **Summary**:

$$
\begin{aligned}
F^2 &= 6[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id}) && \text{sage}\\
FG &= 0 && \text{formula}\\
FH &= 0 && \text{formula}
\end{aligned}
$$

3.13. G. **Summary**:

$$
\begin{aligned}
G^2 &= 264[\mathrm{pt}]_1\otimes[\mathrm{pt}]_2\otimes[\mathrm{pt}]_3\otimes[\mathrm{pt}]_4(\mathrm{id}) && \text{formula}\\
GH &= 0 && \text{formula}
\end{aligned}
$$

(3.13.1)

$$\begin{aligned}
G^2 &= \left(\sum_{j=1}^{22} \sum_{1,(23)} [\mathrm{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4(\mathrm{id}) \right)^2 \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{1,(23)} \sum_{a,(bc)} [\mathrm{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4(\mathrm{id}) [\mathrm{pt}]_a \otimes (e_k)_b \otimes (e_k^\vee)_c \otimes 1_d(\mathrm{id}) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{1,(23)} [\mathrm{pt}]_1 \otimes (e_j)_2 \otimes (e_j^\vee)_3 \otimes 1_4(\mathrm{id}) 1_1 \otimes (e_k)_2 \otimes (e_k^\vee)_3 \otimes [\mathrm{pt}]_4(\mathrm{id}) \\
&= \sum_{j=1}^{22} \sum_{k=1}^{22} \sum_{1,(23)} [\mathrm{pt}]_1 \otimes (e_j e_k)_2 \otimes (e_j^\vee e_k^\vee)_3 \otimes [\mathrm{pt}]_4(\mathrm{id}) \\
&= \sum_{1,(23)} 22 [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id}) \\
&= 12 \cdot 22 [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id})
\end{aligned}$$

since $\alpha_1 \otimes \beta_1 \otimes \gamma_1 \otimes \delta_1(\mathrm{id}) \alpha_2 \otimes \beta_2 \otimes \gamma_2 \otimes \delta_2(\mathrm{id}) = \alpha_1\alpha_2 \otimes \beta_1\beta_2 \otimes \gamma_1\gamma_2 \otimes \delta_1\delta_2(\mathrm{id})$. The second to last line in the equation follows as before.

3.14. H. **Summary**:

$$H^2 = 1584 [\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id}) \qquad \text{formula}$$

(3.14.1)

$$
\begin{aligned}
H^2 &= \left(\sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(12)(34)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4(\mathrm{id})\right)^2 \\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22}\sum_{(12)(34)}\sum_{(ab)(cd)} (e_j)_1 \otimes (e_j^\vee)_2 \otimes (e_k)_3 \otimes (e_k^\vee)_4(\mathrm{id})(e_u)_a \otimes (e_u^\vee)_b \otimes (e_v)_c \otimes (e_v^\vee)_d(\mathrm{id}) \\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22}\sum_{(12)(34)} \big((e_je_u)_1 \otimes (e_j^\vee e_u^\vee)_2 \otimes (e_ke_v)_3 \otimes (e_k^\vee e_v^\vee)(\mathrm{id}) \\
&+ (e_je_u)_1 \otimes (e_j^\vee e_v)_2 \otimes (e_ke_u^\vee)_3 \otimes (e_k^\vee e_v^\vee)(\mathrm{id}) \\
&+ (e_je_u)_1 \otimes (e_j^\vee e_v^\vee)_2 \otimes (e_ke_v)_3 \otimes (e_k^\vee e_u^\vee)(\mathrm{id})\big) \\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22}\sum_{(12)(34)} (a_{ju}a_{ju}^\vee a_{kv}a_{kv}^\vee + a_{ju}a_{kv}^\vee \delta_{j=v}\delta_{k=u} + a_{ju}a_{jv}^\vee a_{kv}a_{ku}^\vee) \\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(12)(34)} a_{jk}a_{kj}^\vee + \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22}\sum_{(12)(34)} (a_{ju}a_{ju}^\vee a_{kv}a_{kv}^\vee + a_{ju}a_{jv}^\vee a_{kv}a_{ku}^\vee) \\
&= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{(12)(34)} a_{jk}a_{kj}^\vee + \sum_{(12)(34)}\left(\sum_{j=1}^{22}\sum_{u=1}^{22} a_{ju}a_{ju}^\vee\right)^2 + \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22}\sum_{(12)(34)} a_{ju}a_{jv}^\vee a_{kv}a_{ku}^\vee \\
&= \sum_{(12)(34)} \big(22 + 22^2 + 22\big) \\
&= 3 \cdot (2 \cdot 22 + 22^2)[\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id}) \\
&= 1584[\mathrm{pt}]_1 \otimes [\mathrm{pt}]_2 \otimes [\mathrm{pt}]_3 \otimes [\mathrm{pt}]_4(\mathrm{id})
\end{aligned}
$$

since

$$\sum_{j=1}^{22}\sum_{k=1}^{22} a_{jk}a_{kj}^\vee = \operatorname{Tr} I_{22} = 22$$

and

$$\sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22} a_{ju}a_{jv}^{\vee}a_{kv}a_{ku}^{\vee} = \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22}(e_j,e_u)(e_j^{\vee},e_v^{\vee})(e_k,e_v)(e_k^{\vee},e_u^{\vee})$$
$$= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22}\sum_{v=1}^{22} r_{ju}t_{jv}r_{kv}t_{ku}$$
$$= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22} r_{ju}t_{ku}\left(\sum_{v=1}^{22} t_{jv}r_{vk}\right)$$
$$= \sum_{j=1}^{22}\sum_{k=1}^{22}\sum_{u=1}^{22} r_{ju}t_{ku}\delta_{j=k}$$
$$= \sum_{j=1}^{22}\sum_{u=1}^{22} r_{ju}t_{ju}$$
$$= 22$$

4. Hodge classes of $S^{[4]}$

We include here the details to the proofs of the statements in [BJ11, §2.7].

(4.1) $c_2(S^{[4]}) = 3Z + 33Y - W$, *cf.* [BJ11, Lemma 2.9].

We had

$$c_2(S^{[4]}) = \left(-\frac{4}{9}u - \frac{4}{27}v\right)W + \left(u - \frac{21}{4}\right)X + (v+42)Y - \frac{v}{3}Z$$

This yields:

$$
\begin{aligned}
945300 = \theta^2 \mathrm{c}_2(S^{[4]})^2 &= \frac{1117}{2}(v+42)^2 - 16(v+42)(4u-21) + \frac{776}{27}(v+42)(3u+v) \\
&- \frac{2365}{3}(v+42)v + \frac{611}{4}(4u-21)^2 - \frac{848}{27}(4u-21)(3u+v) \\
&+ \frac{220}{3}(4u-21)v + \frac{23488}{243}(3u+v)^2 - \frac{5720}{81}(3u+v)v + \frac{19525}{18}v^2 \\
&= \frac{79300}{27}u^2 + \frac{45200}{81}uv + \frac{221050}{243}v^2 - 22750u + \frac{43400}{3}v + \frac{4266675}{4} \\
-246600 = \theta\delta^2 \mathrm{c}_2(S^{[4]}) &= -95(v+42)^2 + 52(v+42)(4u-21) - \frac{848}{9}(v+42)(3u+v) \\
&+ \frac{814}{3}(v+42)v - \frac{139}{2}(4u-21)^2 + \frac{992}{9}(4u-21)(3u+v) \\
&- \frac{748}{3}(4u-21)v - \frac{7552}{81}(3u+v)^2 + \frac{6512}{27}(3u+v)v - \frac{2519}{9}v^2 \\
&= -\frac{5656}{9}u^2 - \frac{12608}{27}uv - \frac{4036}{81}v^2 + 1596u + 1288v - \frac{488187}{2} \\
177552 = \delta^4 \mathrm{c}_2(S^{[4]}) &= 42(v+42)^2 - 48(v+42)(4u-21) + 96(v+42)(3u+v) \\
&- 220(v+42)v + 87(4u-21)^2 - 192(4u-21)(3u+v) \\
&+ 352(4u-21)v + \frac{1280}{9}(3u+v)^2 - 352(3u+v)v + \frac{638}{3}v^2 \\
&= 368u^2 + \frac{1600}{3}uv - \frac{712}{9}v^2 + 1512u - 4032v + 154791
\end{aligned}
$$

There are two solutions, $(u,v) = (\frac{21}{4}, -9), (\frac{497}{116}, -\frac{285}{29})$. Finally, we compute

$$
\begin{aligned}
1992240 = \mathrm{c}_2(S^{[4]})^4 = &\frac{113482}{729}u^4 + \frac{361912}{2187}u^3v + \frac{273944}{2187}u^2v^2 + \frac{67072}{19683}uv^3 - \frac{242858}{81}u^3 \\
&+ \frac{556552}{59049}v^4 - \frac{284438}{81}u^2v - \frac{356636}{243}uv^2 + \frac{146951}{4}u^2 + \frac{1193920}{2187}v^3 \\
&+ \frac{139013}{6}uv + \frac{476329}{18}v^2 - \frac{1416933}{8}u + \frac{2028159}{8}v + \frac{447500781}{128}
\end{aligned}
$$

Which rules out $(u,v) = (\frac{497}{116}, -\frac{285}{29})$.

(4.2) The class $\alpha = X - 3Y + Z$ intersects trivially with

$$\lambda^4\theta, \lambda^4\mathrm{c}_2(X), \lambda^2\theta^2, \lambda^2\theta\mathrm{c}_2(X), \lambda^2\mathrm{c}_2(X)^2, \theta^3, \theta^2\mathrm{c}_2(X), \theta\mathrm{c}_2(X)^2, \mathrm{c}_2(X)^3$$

and

$$
\begin{aligned}
\alpha^2\theta^2 &= 9450 \\
\alpha^2\theta\mathrm{c}_2(X) &= 14148 \\
\alpha^2\mathrm{c}_2(X)^2 &= 21168
\end{aligned}
$$

cf. [BJ11, Corollary 2.11]

We'll demonstrate the last 3 identities. Since

$$\theta = -\frac{1}{2}W - \frac{1}{3}X + \frac{45}{2}Y + \frac{13}{6}Z$$

cf. [BJ11, §1.10] , we have

$$\begin{aligned}
\alpha^2 &= -3G + 30D + 42F + 3H + 6E \\
\alpha\theta &= 88D + 8E - 27C + 3B - 88F + 20G + 4H \\
\alpha \mathrm{c}_2(S^{[4]}) &= -54C + 132D + 6B + 12E + 30G + 6H - 132F \\
\theta^2 &= -8E + \frac{19}{2}H + \frac{215}{2}G - 64D - \frac{33}{4}A - \frac{97}{4}B + 1117F - \frac{873}{4}C \\
\theta \mathrm{c}_2(S^{[4]}) &= -\frac{27}{2}A - \frac{747}{2}C - \frac{83}{2}B - 8E + 1630F + 153G + 13H - 48D \\
\mathrm{c}_2(S^{[4]})^2 &= 18H - 8E - 69B - 8D + 218G - 21A - 621C + 2380F
\end{aligned}$$

and the pairwise products are easily computed.

5. Fujiki Constants of X

For the sake of completeness we describe the computation of the integrals

$$\int_{S^{[n]}} \delta^k \mathrm{c}_\mu(S^{[n]})$$

for $S = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ and $\delta = \det \mathcal{O}^{[n]}$by toric localization, *cf.* [BJ11, Section 2.1].

(5.1) First consider $S = \mathbb{A}^2$, which has an action by $G = \mathbb{G}_m^2$ via $(x, y) \mapsto (\lambda x, \mu y)$ where λ, μ are the characters obtained by projecting to each factor. The only fixed point is the origin $(0, 0)$. G also acts on $(\mathbb{A}^2)^{[n]}$; fixed points are length n subschemes Z fixed by G. Thus, they must be supported on a fixed point (i.e. the origin), and the ideal $I_Z \subset A = \mathbb{C}[x, y]$ must be generated by monomials. I_Z is determined by the monomials $x^a y^b$ left out of the ideal, which form a Young tableau with n boxes. Given such a Young tableau in the upper right quadrant, let $(i, b_i - 1)$ for $0 \le i \le n-1$ be the extremal boxes, so b_i is the height of the ith column. A partition μ of n uniquely determines a Young tableau by arranging μ_i columns of height i in descending order.

(5.2) For a space X with an action by G with isolated fixed points, Bott localization implies

$$\int_X \varphi = \sum_{p \in X^G} \int \frac{i_p^* \varphi}{\mathrm{c}_{\mathrm{top}}(T_p X)}$$

where $\varphi \in H_G^*(X)$, $i_p^* : H_G^*(X) \to H_G^*(X^G) \cong H^*(X^G) \otimes H_G^*([\mathrm{pt}])$ is the pull-back to a fixed point $p \in X^G$. The Chern class is the equivariant chern class of the G representation $T_p X$.

(5.3) For a partition μ representing a fixed point p_μ of $X = (\mathbb{A}^2)^{[n]}$, the Chern polynomial is [ES87, Lemma 3.2]

$$C(\mu; \alpha, \beta) := \sum_i t^t \mathrm{c}_{2n-i}(T_{p_\mu} X) = \prod_{1 \le i \le j \le n} \prod_{s=b_j}^{b_{j-1}-1} (t + (i-j-1)\alpha + (b_{i-1} - s - 1)\beta)(t + (j-i)\alpha + (s - b_{i-1})\beta) \tag{5.1}$$

where $\alpha = \mathrm{c}_1(\lambda), \beta = \mathrm{c}_1(\mu)$. $\mathcal{O}^{[n]}$ restricted to a point of $\mathbb{A}^{[n]}$ corresponding to a subscheme Z is canonically $\mathcal{O}_Z$, so setting $f = \mathrm{c}_1(\mathcal{O}^{[n]})$,

$$Z(\mu; \alpha, \beta) := i_{p_\mu}^* f = \sum_{i=0}^{n} \sum_{j=0}^{b_i - 1} i\alpha + j\beta \tag{5.2}$$

(5.4) For $S = \mathbb{P}^2$, let $\mathbb{G}_m^2$ act on $[x, y, z]$ via $[\lambda x, \mu y, z]$. There are three fixed points $p_0 = [0,0,1], p_1 = [0,1,0]$, $p_2 = [1,0,0]$, and a length n subscheme Z of $\mathbb{P}^2$ will consist of a length n_i subscheme Z_i at p_i with $\sum n_i = n$. The tangent space at such a point is canonically

$$T_Z(\mathbb{P}^2)^{[n]} = \bigoplus_i T_{Z_i}(\mathbb{P}^2)^{[n_i]}$$

Note that at any point $[Z] \in (\mathbb{P}^2)^{[n]}$ corresponding to a subscheme Z supported at p_i, there is a $\mathbb{G}_m^2$-stable Zariski neighborhood isomorphic to $\mathbb{A}^{[n]}$ with torus action via $(\lambda x, \mu y)$, $(\lambda\mu^{-1}x, \mu^{-1}y)$, $(\mu\lambda^{-1}x, \lambda^{-1}y)$ for $i = 0, 1, 2$ respectively.

(5.5) A 3-vector partition $\underline{\mu}$ of n will be three partitions (μ_1, μ_2, μ_3) such that $|\mu_1| + |\mu_2| + |\mu_3| = n$; 3-vector partitions of n classify fixed points $p_{\underline{\mu}}$ of $X = (\mathbb{P}^2)^{[n]}$. By the above, the tangent space at $p_{\underline{\mu}}$ has Chern polynomial

$$\sum t^i C_{2n-i}(\underline{\mu}; \alpha, \beta) = C(\mu_1; \alpha, \beta) C(\mu_2; \alpha - \beta, -\beta) C(\mu_3; \beta - \alpha, -\alpha) \tag{5.3}$$

Define $C_i(\underline{\mu}; \alpha, \beta) = \mathrm{c}_i(T_{p_{\underline{\mu}}} X)$. Also,

$$Z(\underline{\mu}; \alpha, \beta) := i_{p_{\underline{\mu}}}^* f = Z(\mu_1; \alpha, \beta) + Z(\mu_2; \alpha - \beta, -\beta) + Z(\mu_3; \beta - \alpha, -\alpha) \tag{5.4}$$

The final answer is then, for $X = (\mathbb{P}^2)^{[n]}$

$$\int_X f^{2n - \sum_i k_i} \prod_i \mathrm{c}_{k_i}(TX) = \sum_{\underline{\mu}, |\underline{\mu}| = n} \frac{Z(\underline{\mu}; \alpha, \beta)^{2n - \sum_i k_i} \prod_i C_{k_i}(\underline{\mu}; \alpha, \beta)}{C_{2n}(\underline{\mu}; \alpha, \beta)} \tag{5.5}$$

(5.6) Let $\mathbb{G}_m^2$ act on $S = \mathbb{P}^1 \times \mathbb{P}^1$ via $[\lambda x_1, y_1] \times [\mu x_2, y_2]$. The fixed points are classified by 4-vector partitions $\underline{\mu}$. Now we have

$$\sum t^i C'_{2n-i}(\underline{\mu}; \alpha, \beta) = C(\mu_1; \alpha, \beta) C(\mu_2; -\alpha, \beta) C(\mu_3; \alpha, -\beta) C(\mu_4; -\alpha, -\beta) \tag{5.6}$$

Also,

$$Z'(\underline{\mu}; \alpha, \beta) := i_{p_{\underline{\mu}}}^* f = Z(\mu_1; \alpha, \beta) + Z(\mu_2; -\alpha, \beta) + Z(\mu_3; \alpha, -\beta) + Z(\mu_4; -\alpha, -\beta) \tag{5.7}$$

The final answer is then once again

$$\int_{(\mathbb{P}^1 \times \mathbb{P}^1)^{[n]}} f^{2n - \sum_i k_i} \prod_i \mathrm{c}_{k_i}(TX) = \sum_{\underline{\mu}, |\underline{\mu}| = n} \frac{Z'(\underline{\mu}; \alpha, \beta)^{2n - \sum_i k_i} \prod_i C'_{k_i}(\underline{\mu}; \alpha, \beta)}{C'_{2n}(\underline{\mu}; \alpha, \beta)} \tag{5.8}$$

(5.7) Let Φ be the universal genus, *cf.* [BJ11, §2.1]. We have

$$\sum_{n \geq 0} z^n \int_{S^{[n]}} \exp \det(\mathcal{O}^{[n]}) \Phi(S^{[n]}) = \mathbf{A}(z)^{\mathrm{c}_1(S)^2} \mathbf{B}(z)^{\mathrm{c}_2(S)}$$

(5.8) The universal genus for vanishing odd Chern classes up to degree 8 is:

$$
\begin{aligned}
\Phi = 1 &+ (a_1^2 - 2a_2)\mathrm{c}_2 \\
&+ (a_2^2 - 2a_1a_3 + 2a_4)\mathrm{c}_2^2 \\
&+ (a_1^4 - 4a_1^2a_2 + 2a_2^2 + 4a_1a_3 - 4a_4)\mathrm{c}_4 \\
&+ (a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_6)\mathrm{c}_2^3 \\
&+ (a_1^2a_2^2 - 2a_1^3a_3 - 2a_2^3 + 4a_1a_2a_3 + 2a_1^2a_4 - 3a_3^2 + 2a_2a_4 - 6a_1a_5 + 6a_6)\mathrm{c}_2\mathrm{c}_4 \\
&+ (a_1^6 - 6a_1^4a_2 + 9a_1^2a_2^3 + 6a_1^3a_3 - 2a_2^3 - 12a_1a_2a_3 - 6a_1^2a_4 + 3a_3^2 + 6a_2a_4 + 6a_1a_5 - 6a_6)\mathrm{c}_6 \\
&+ (a_4^2 - 2a_3a_5 + 2a_2a_6 - 2a_1a_7 + 2a_8)\mathrm{c}_2^4 \\
&+ (a_1^2a_3^2 - 2a_1^2a_2a_4 + 2a_1^3a_5 - 2a_2a_3^2 + 4a_2^2a_4 - 4a_1a_2a_5 \\
&- 2a_1^2a_6 - 4a_4^2 + 8a_3a_5 - 4a_2a_6 + 8a_1a_7 - 8a_8)\mathrm{c}_2^2\mathrm{c}_4 \\
&+ (a_1^4a_2^2 - 2a_1^5a_3 - 4a_1^2a_2^3 + 8a_1^3a_2a_3 + 2a_1^4a_4 + 2a_2^4 - 9a_1^2a_3^2 \\
&- 6a_1^2a_2a_4 - 2a_1^3a_5 + 2a_2a_3^2 - 4a_2^2a_4 + 16a_1a_3a_4 + 4a_1a_2a_5 \\
&+ 2a_1^2a_6 - 4a_4^2 - 8a_3a_5 + 4a_2a_6 - 8a_1a_7 + 8a_8)\mathrm{c}_2\mathrm{c}_6 \\
&+ (a_2^4 - 4a_1a_2^2a_3 + 2a_1^2a_3^2 + 4a_1^2a_2a_4 - 4a_1^3a_5 + 4a_2a_3^2 - 4a_2^2a_4 - 8a_1a_3a_4 \\
&+ 8a_1a_2a_5 + 4a_1^2a_6 + 6a_4^2 - 4a_3a_5 - 4a_2a_6 - 4a_1a_7 + 4a_8)\mathrm{c}_4^2 \\
&+ (a_1^8 - 8a_1^6a_2 + 20a_1^4a_2^2 + 8a_1^5a_3 - 16a_1^2a_2^3 - 32a_1^3a_2a_3 - 8a_1^4a_4 + 2a_2^4 \\
&+ 24a_1a_2^2a_3 + 12a_1^2a_3^2 + 24a_1^2a_2a_4 + 8a_1^3a_5 - 8a_2a_3^2 - 8a_2^2a_4 - 16a_1a_3a_4 \\
&- 16a_1a_2a_5 - 8a_1^2a_6 + 4a_4^2 + 8a_3a_5 + 8a_2a_6 + 8a_1a_7 - 8a_8)\mathrm{c}_8
\end{aligned}
$$

(5.9) By localization, we compute:

$$\begin{aligned}
\mathbf{A}(z) = 1 &+ a_2 z + \left(-a_1^3 + 3a_1^2a_2 + \frac{1}{4}a_1^2 + a_1a_2 - \frac{9}{2}a_2^2 + a_1a_3 + \frac{1}{6}a_1 - \frac{3}{2}a_2 + 3a_3 - 10a_4 - \frac{1}{48}\right) z^2 \\
&+ \left(-3a_1^5 + 4a_1^4a_2 + \frac{5}{2}a_1^4 + 5a_1^3a_2 - \frac{23}{3}a_1^2a_2^2 - \frac{8}{3}a_1^3a_3 + 5a_1^3 - \frac{87}{4}a_1^2a_2 - 21a_1a_2^2 + \frac{179}{6}a_2^3\right. \\
&+ 62a_1^2a_3 - \frac{125}{3}a_1a_2a_3 - \frac{160}{3}a_1^2a_4 - \frac{19}{12}a_1^2 - \frac{22}{3}a_1a_2 + 39a_2^2 - 19a_1a_3 - 44a_2a_3 - \frac{56}{3}a_3^2 \\
&\left.-11a_1a_4 + 138a_2a_4 - \frac{64}{3}a_1a_5 - \frac{17}{40}a_1 + \frac{223}{48}a_2 - \frac{15}{2}a_3 + 37a_4 - 55a_5 + \frac{364}{3}a_6 + \frac{19}{240}\right) z^3 \\
&+ \left(-7a_1^7 + 7a_1^6a_2 + 12a_1^6 + 15a_1^5a_2 - \frac{37}{2}a_1^4a_2^2 - 9a_1^5a_3 + \frac{159}{4}a_1^5 - \frac{507}{4}a_1^4a_2 - \frac{281}{2}a_1^3a_2^2\right. \\
&+ \frac{827}{6}a_1^2a_2^3 + 370a_1^4a_3 - \frac{623}{3}a_1^3a_2a_3 - 150a_1^4a_4 - \frac{2461}{96}a_1^4 - \frac{451}{4}a_1^3a_2 + \frac{3563}{8}a_1^2a_2^2 + \frac{353}{2}a_1a_2^3 \\
&- \frac{4663}{24}a_2^4 - \frac{1007}{4}a_1^3a_3 - 719a_1^2a_2a_3 + \frac{2147}{6}a_1a_2^2a_3 - \frac{21}{2}a_1^2a_3^2 - 67a_1^3a_4 + \frac{1799}{3}a_1^2a_2a_4 \\
&+ 2a_1^3a_5 - \frac{1403}{80}a_1^3 + \frac{6535}{48}a_1^2a_2 + \frac{717}{4}a_1a_2^2 - \frac{2495}{4}a_2^3 - \frac{4045}{12}a_1^2a_3 + \frac{1501}{2}a_1a_2a_3 + \frac{1153}{2}a_2^2a_3 \\
&- 840a_1a_3^2 + \frac{1027}{3}a_2a_3^2 + \frac{1033}{2}a_1^2a_4 + 741a_1a_2a_4 - 1611a_2^2a_4 + 686a_1a_3a_4 - 1147a_1^2a_5 \\
&+ \frac{1424}{3}a_1a_2a_5 + 973a_1^2a_6 + \frac{18001}{2880}a_1^2 + \frac{2291}{80}a_1a_2 - \frac{6689}{32}a_2^2 + \frac{1509}{16}a_1a_3 + \frac{793}{3}a_2a_3 \\
&+ 235a_3^2 + \frac{322}{3}a_1a_4 - 1597a_2a_4 + 305a_3a_4 - 782a_4^2 + 431a_1a_5 + 1096a_2a_5 + 633a_3a_5 \\
&+ 127a_1a_6 - \frac{7148}{3}a_2a_6 + 383a_1a_7 + \frac{10061}{10080}a_1 - \frac{7219}{480}a_2 + \frac{1403}{80}a_3 - \frac{1037}{8}a_4 + \frac{545}{3}a_5 \\
&\left.-731a_6 + 889a_7 - 1608a_8 - \frac{37861}{161280}\right) z^4
\end{aligned}$$

(5.10)

$$
\begin{aligned}
\mathbf{B}(z) = 1 &+ \left(a_1^2 - 2a_2\right) z + \left(2a_1^4 - 8a_1^2a_2 - \frac{5}{4}a_1^2 + \frac{31}{2}a_2^2 - 15a_1a_3 + \frac{5}{2}a_2 + 15a_4 + \frac{1}{48}\right) z^2 \\
&+ \Bigg(3a_1^6 - 18a_1^4a_2 - \frac{45}{4}a_1^4 + \frac{461}{6}a_1^2a_2^2 - \frac{245}{3}a_1^3a_3 + 45a_1^2a_2 - \frac{317}{3}a_2^3 + \frac{490}{3}a_1a_2a_3 \\
&+ \frac{245}{3}a_1^2a_4 + \frac{121}{48}a_1^2 - 65a_2^2 + 40a_1a_3 + \frac{280}{3}a_3^2 - 350a_2a_4 + \frac{560}{3}a_1a_5 - \frac{121}{24}a_2 - 40a_4 \\
&- \frac{560}{3}a_6 - \frac{1}{18}\Bigg) z^3 \\
&+ \Bigg(5a_1^8 - 40a_1^6a_2 - 50a_1^6 + \frac{1565}{6}a_1^4a_2^2 - \frac{845}{3}a_1^5a_3 + 300a_1^4a_2 - \frac{2170}{3}a_1^2a_2^3 + \frac{3380}{3}a_1^3a_2a_3 \\
&+ \frac{845}{3}a_1^4a_4 + \frac{4207}{96}a_1^4 - \frac{7375}{8}a_1^2a_2^2 + \frac{19265}{24}a_2^4 + \frac{2575}{4}a_1^3a_3 - \frac{10585}{6}a_1a_2^2a_3 + \frac{7955}{6}a_1^2a_3^2 \\
&- \frac{7510}{3}a_1^2a_2a_4 + \frac{4130}{3}a_1^3a_5 - \frac{4207}{24}a_1^2a_2 + \frac{4175}{4}a_2^3 - \frac{2575}{2}a_1a_2a_3 - \frac{4130}{3}a_2a_3^2 - \frac{2575}{4}a_1^2a_4 \\
&+ \frac{9035}{2}a_2^2a_4 - 1275a_1a_3a_4 - \frac{8260}{3}a_1a_2a_5 - \frac{4130}{3}a_1^2a_6 - \frac{3455}{576}a_1^2 + \frac{21883}{96}a_2^2 - \frac{1685}{16}a_1a_3 \\
&- \frac{735}{2}a_3^2 + \frac{4045}{2}a_2a_4 + 1950a_4^2 - 735a_1a_5 - 2625a_3a_5 + \frac{16135}{3}a_2a_6 - 2625a_1a_7 + \frac{3455}{288}a_2 \\
&+ \frac{1685}{16}a_4 + 735a_6 + 2625a_8 + \frac{625}{4608}\Bigg) z^4
\end{aligned}
$$

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