CM points have everywhere good reduction

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Abstract

We prove that for every Shimura variety S, there is an integral model $\mathcal{S}$ such that all CM points of S have good reduction with respect to $\mathcal{S}$. In other words, every CM point is contained in $\mathcal{S}(\overline{\mathbb{Z}})$. This follows from a stronger local result wherein we characterize the points of S with potentially-good reduction (with respect to some auxiliary prime ℓ) as being those that extend to integral points of $\mathcal{S}$.

1 Introduction

The Néron-Ogg-Shafarevich criterion [8] says that for an abelian variety over $\mathbb{Q}$, one can detect (potentially) good reduction at a prime p, by looking at whether the representation on its ℓ-adic Tate module is (potentially) unramified, for $\ell \neq p$. One can reinterpert this result as being purely about the moduli space of (principally polarized) abelian varieties $\mathcal{A}_g$ and the $\mathbb{Z}_\ell$-adic local system $\mathcal{L}$ on it corresponding to the universal Tate module: *A point $x \in \mathcal{A}_g(\mathbb{Q}_p)$ is in the image of $\mathcal{A}_g(\mathbb{Z}_p)$ iff the $\mathrm{Gal}_{\mathbb{Q}_p}$-representation $\mathcal{L}_x$ is unramified.*

Our goal is to generalize this result to Shimura varieties which are not of abelian-type. One case of particular interest is that of CM points, and we record the following theorem:

Theorem 1. *Let $S = S_K(G, X)$ be a Shimura variety over $E = E(G, X)$. There is an integral model $\mathcal{S}$ over $\mathcal{O}_E$ for S such that every CM point of $S(\overline{\mathbb{Q}})$ extends to a point of $\mathcal{S}(\overline{\mathbb{Z}})$.*

Note that in [7] this was established away from finitely many 'bad' primes, and the contribution of this note is to handle precisely that bad set.

This result is a consequence of the more precise result below. Let V be a faithful $\mathbb{Q}$-representation of G^{ad}, and ${}_{\mathrm{et}}V_\ell$ the corresponding ℓ-adic local system. Let $v \nmid \ell$ be a finite place of E. For a local field $F \subset \overline{E_v}$ We say a point $x \in S(F)$ is *ℓ-potentially unramified* if the Galois representation ${}_{\mathrm{et}}V_{\ell,x}$ is potentially unramified. Let $S_v^{\ell-pun} \subset S(\overline{E}_v)$ denote the set of potentially unramified points.

Theorem 2. *Let $S = S_K(G, X)$ be a Shimura variety over $E = E(G, X)$, and fix a prime number ℓ. There exists an integral model $\mathcal{S}$ over $\mathcal{O}_E$ such that for each finite place $v \nmid \ell$ of E, we have that*

$$S_v^{\ell-pun} = \mathcal{S}(\mathcal{O}_{\overline{E}_v}).$$

Moreover, if ${}_{\mathrm{et}}V_\ell$ has torsion-free monodromy group, one may pick $\mathcal{S}$ so that ${}_{\mathrm{et}}V_\ell$ extends to $\mathcal{S}$.

Remark 1.1.

1. If the monodromy group is not torsion-free then by our arguments one may still construct a Deligne-Mumford stack $\mathcal{S}$ compactifying S so that ${}_{\mathrm{et}}V_\ell$ extends.
2. It seems reasonable to conjecture that there is a model which works independently of ℓ—indeed, this is true for 'good' primes v, as we show below. It is the finitely many 'bad' primes at which S does not have a log-smooth model that we cannot show this.

3. Indeed, even for proper Shimura varieties S, at 'bad' finite places v we cannot show that the ${}_{\text{et}}V_{\ell,x}$ are potentially unramified for all $x \in S(E_v)$. Note that this would not follow simply from the existence of an appropriate motivic family over S, since such a family might a-priori have bad reduction at v.

4. The proof of Theorem 2 applies verbatim to any smooth variety S/E with a $\mathbb{Z}_\ell$-local system $\mathcal{L}$ which is non-extendable (or maximal) in the sense of [3, Def 3.1], and such that $\mathcal{L}_x$ is almost everywhere unramified for some (hence any) $x \in S(\overline{E})$. In particular, the latter condition holds for any local system of geometric origin.

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2 Background

Let $\mathcal{O}$ be a mixed characteristic p DVR with fraction field K and residue field k. Let $\mathcal{X}, \mathcal{X}'/\mathcal{O}$ be reduced separated schemes which are flat and finite type over $\mathcal{O}$. For an open subset $U \subset \mathcal{X}$, a U-admissible modification $\mathcal{X}' \to \mathcal{X}$ (over $\mathcal{O}$) is proper morphism which is an isomorphism over U. If U is the generic fiber, we simply refer to admissible modifications.

Let X/K be a variety. A partial compactification for X over $\mathcal{O}$ is a reduced separated scheme $\mathcal{X}$, flat and finite type over $\mathcal{O}$ and a dense open embedding $X \subset \mathcal{X}$ of $\mathcal{O}$-schemes. If $\mathcal{X}_K \cong X$ we say $\mathcal{X}$ is an integral model of X; if $\mathcal{X} \to \operatorname{Spec} \mathcal{O}$ is proper, we say $\mathcal{X}$ is a compactification over $\mathcal{O}$.

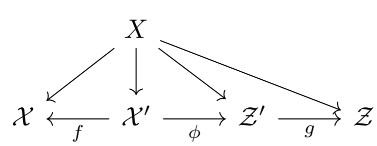
If R is a Dedekind domain with fraction field K, we say that a finite type $\mathcal{X}/R$ is an integral model of X if it is over every prime ideal of R.

2.1 Chow's lemma

We shall require the following version of Chow's lemma:

Lemma 2.1. *Let X/K be a quasiprojective variety and $\mathcal{X}$ a compactification over $\mathcal{O}$. Then there exists a flat projective $f' : \mathcal{X}' \to \operatorname{Spec} \mathcal{O}$ and an X-admissible modification $\psi : \mathcal{X}' \to \mathcal{X}$ over $\mathcal{O}$.*

Proof. This is essentially [9, Tag 088R]. In the notation of that lemma, we let $Y = \operatorname{Spec} \mathcal{O}, U = X$. Applying the lemma, we obtain a commutative diagram of $\operatorname{Spec} \mathcal{O}$ schemes:



where the maps from X are open immersions, f is an X-admissible modification, ϕ is an open immersion, g is an X-admissible modification and $\mathcal{Z}$ is a projective $\operatorname{Spec} \mathcal{O}$ scheme.

Since blowups are projective, it follows that $\mathcal{Z}'$ is projective over $\operatorname{Spec} \mathcal{O}$, and since $\mathcal{X}'$ is proper and ϕ is an open immersion it follows that ϕ is an isomorphism onto a closed subscheme of $\mathcal{Z}'$, and hence $\mathcal{X}'$ is also projective over $\operatorname{Spec} \mathcal{O}$.

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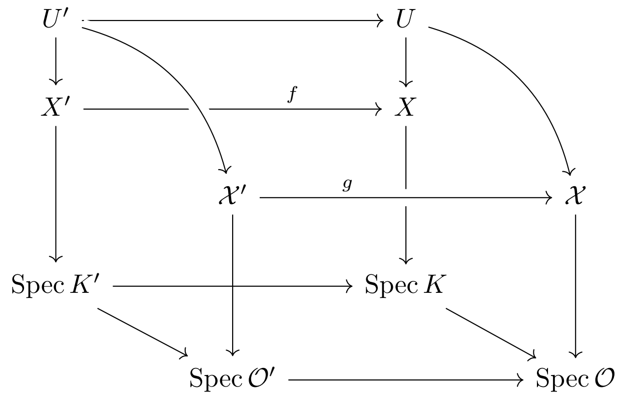
2.2 Integral models

Lemma 2.2. *Let $\mathcal{O}'/\mathcal{O}$ be an étale extension of DVRs, and X/K, X'/K' normal quasiprojective varieties with an étale morphism $f : X' \to X$ over $\operatorname{Spec} K' \to \operatorname{Spec} K$. Let $U \subset X$ be the subset over which f is finite étale and $U' := f^{-1}(U)$. Then for any compactification $\mathcal{X}'_0$ of X' over $\mathcal{O}'$, there is an U'-admissible modification $\mathcal{X}' \to \mathcal{X}'_0$ and a projective compactification $\mathcal{X}$ of U over $\mathcal{O}$ fitting into the commutative diagram below.*



Moreover, g is a quotient by a finite group action.

Proof. Let X''/K'' and $\mathcal{X}''_0/\mathcal{O}''$ be the normalizations of X' and $\mathcal{X}'_0$ in the Galois closure of the function field extension of $X' \to X$, G the Galois group, and U'' the preimage of U'. By Lemma 2.1, up to replacing $\mathcal{X}''_0$ with a U''-admissible modification we may assume $\mathcal{X}''_0$ is quasiprojective. Let $\mathcal{X}''^G_0$ be the G-fold fiber product of $\mathcal{X}''_0$ over $\mathcal{O}''$ equipped with the G-action by permuting factors. Letting $\mathcal{X}'' \subset \mathcal{X}''^G_0$ be the closure of the natural G-equivariant embedding $U'' \to \mathcal{X}''^G_0$, any projection $\mathcal{X}'' \to \mathcal{X}''_0$ is a U''-admissible modification. Letting $\mathcal{X}$ (resp. $\mathcal{X}'$) be the quotient of $\mathcal{X}''$ by G (resp. the Galois group of X''/X'), we obtain the required $\mathcal{X}$ (resp. $\mathcal{X}'$). ☐

Corollary 2.3. *Let $\hat{\mathcal{O}}$ be the completion of $\mathcal{O}$ with fraction field $\hat{K}$, X/K a quasiprojective variety, and $\hat{\mathcal{X}}$ an integral model of $X_{\hat{K}}$. Then there exists a quasiprojective model $\mathcal{X}$ of X and an admissible morphism $\mathcal{X}_{\hat{\mathcal{O}}} \to \hat{\mathcal{X}}$.*

Proof. By [10, Claim 3.1.3.1] the model $\hat{\mathcal{X}}$ descends canonically to a model $\mathcal{X}'_0$ of $X_{K'}$ defined over an étale DVR extension $\mathcal{O}'/\mathcal{O}$. Now apply Lemma 2.2. ☐

Now assume that K is a local field. Let $D \subset X$ be a closed subscheme. We let X^{an} denote the Berkovich space [1] associated to X through analytification and let $\mathcal{X}$ denote an integral model of X.

The following is almost certainly well known, but we could not find a reference so provide our own proof:

Lemma 2.4. *Assume $\mathcal{X}/\mathcal{O}$ is proper. For any neighborhood U of D^{an}, there exists an admissible modification $\mathcal{Y} \to \mathcal{X}$ such that if $\mathcal{D}$ is the closure of D in $\mathcal{Y}$, then the complement U^c is contained in the rigid fiber of $\mathcal{Y}\backslash\mathcal{D}$.*

Proof. Setting $\mathcal{X} = \mathcal{X}_0$, we form a tower of models $\pi_{n+1} : \mathcal{X}_{n+1} \to \mathcal{X}$ inductively, by letting D_n be the closure of D in $\mathcal{X}_n$ and blowing up the special fiber $(D_n)_k$. We claim that for large enough n, the model $\mathcal{X}_n$ suffices. To prove this, it is sufficient to work locally on $\mathcal{X}$, so we consider an affine open subset $\operatorname{Spec} R \subset \mathcal{X}$ where $D \cap \operatorname{Spec} R$ has defining ideal $(f_1, \ldots, f_n)$.

Inductively then, $\mathcal{D}_m \cap \pi_m^{-1} \operatorname{Spec} R$ lies in a single affine chart, and is cut out by $(\frac{f_1}{\pi_K^m}, \cdots, \frac{f_n}{\pi_K^m})$, and so a valuation $w \in (\operatorname{Spec} R)^{\mathrm{rig}}$ maps to $\mathcal{D}_m$ iff

$$\max_i |f_i(Q)|_w < |\pi_K^m|.$$

For a sufficiently large n this lies within our tube U by continuity, completing the proof.

☐

2.3 Local systems

We refer to [6, 5] for background on Shimura varieties.

Let (G, X) be a Shimura datum satisfying the axioms in [6], and $S = S_K(G, X)$ be a Shimura variety with reflex field $E = E(G, X)$. Let V be a faithful $\mathbb{Q}$-representation of G^{ad}, and $\mathbb{V} \subset V$ be a K-stable lattice. For a prime number ℓ, we let ${}_{\mathrm{et}}\mathbb{V}_\ell$ denote the corresponding $\mathbb{Z}_\ell$-local system. Furthermore, there exists a natural $\mathbb{Z}$-local system ${}_B\mathbb{V}$ on $S(\mathbb{C})$ underlying a variation of Hodge structures, with proper corresponding period map. In other words, if $\overline{S}$ is a log-smooth compactification of $S = S_K(G, X)$, then V has infinite monodromy around every irreducible component of $D := \overline{S}\backslash S$ (see [4, Theorem 9.5].

We shall require the following lemma:

Lemma 2.5. *Let $\mathcal{L}$ be a local system underlying a VHS on a smooth complex variety X with log-smooth compactification $\overline{X}\backslash X$, and assume the monodromy around each boundary divisor is unipotent. Let $Q \in \overline{X}$ and let $F_1, \ldots, F_m$ be the irreducible divisors of $D := \overline{X}\backslash X$ containing Q. Let $q \in X$ be an infinitesimally nearby point to Q and consider the monodromy elements $\sigma_1, \ldots, \sigma_m \in \pi_1(X, q)$ corresponding to the simple loops around $F_1, \ldots, F_m$. Then there is no nontrivial product $\prod_i \sigma_i^{\mathbb{Z}_{\geqslant 0}}$ which acts trivially on $\mathcal{L}_q$.*

Proof. This follows immediately from [4, Proposition 9.11, Theorem 9.5]. ☐

2.4 Local étale fundamental groups of semistable schemes

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a complete regular local ring of mixed characteristic p, with algebraically closed residue field, and with a regular system of parameters $(x_1, \ldots, x_n, y_1, \ldots, y_m)$. Let $S = R\left[\frac{1}{y_1}, \ldots \frac{1}{y_m}\right]$. Then the maximal prime-to-p Galois étale extension S_0 of S is generated by the prime-to-p roots of the y_i.*

Proof. Let W be a Galois étale extension of S of degree prime-to-p. Let T denote the normal closure of R in W. Now for each $j \in \{1, \ldots, m\}$ we let R_j and T_j denote the localization of R and T at the prime ideals (y_j) and $\mathfrak{y}_j$ respectively, where $\mathfrak{y}_j$ is some prime ideal of T_j sitting above (y_j). Now T_j, R_j are discrete valuation rings. Let e_j denote the ramification degree and let $e = \prod_j e_j$. Now let W' denote the compositum of W and the e'th roots of all the y_i, and let T', T'_j be as before. Then by Abhyankar's lemma [9, Tag 0BRM], T'_j is unramified over $R[y_1^{\frac{1}{e}}, \ldots, y_m^{\frac{1}{e}}]_j$ for all j. By the purity of the branch locus [9, Tag 0BMB] it follows that T' is unramified over $R[y_1^{\frac{1}{e}}, \ldots, y_m^{\frac{1}{e}}]$. This latter ring is complete with algebraically closed residue field, and so $T' = R[y_1^{\frac{1}{e}}, \ldots, y_m^{\frac{1}{e}}]$. The claim is thus proven. ☐

2.5 Descending local systems under finite maps

Proposition 2.7. *Let K be a local field with residue field k, and assume k is perfect. Let $\mathcal{Y}/\mathcal{O}_K$ be finite type, separated, flat, quasi-projective, and assume $\mathcal{Y}$ is normal. Let G be a finite group acting on $\mathcal{Y}$, and let $f : \mathcal{Y} \to \mathcal{X}$ be the quotient. Assume the action of G is free on $Y = \mathcal{Y}_K$.*

Let $\mathcal{L}$ be a $\mathbb{Z}_\ell$-local system on X with torsion free monodromy. If $\mathcal{L}_Y$ extends to $\mathcal{L}_{\mathcal{Y}}$, then $\mathcal{L}$ extends to $\mathcal{X}$.

Proof. Let $y \in \mathcal{Y}$ be a closed point, and $I < G$ the stabilizer of $i_y : \operatorname{Spec} k(y) \to \mathcal{Y}$.

Let $X_{\mathcal{L}}$ be the pro-scheme corresponding to the monodromy of $\mathcal{L}$, with $M := \mathrm{Aut}(\mathcal{Y}_{\mathcal{L}}/\mathcal{Y})$. Let $Y_{\mathcal{L}} := Y \times_X X_{\mathcal{L}}$ and $\mathcal{Y}_{\mathcal{L}}$ the normalization of $\mathcal{Y}$. Note by normality of $\mathcal{Y}$ that $\phi : \mathcal{Y}_{\mathcal{L}} \to \mathcal{Y}$ is Galois. Therefore $M \times G$ acts on $\mathcal{Y}_{\mathcal{L}}$, and $M \times I$ acts on $\phi^{-1}(\overline{y})$, while M acts simply transitively on this set. Since there are no non-trivial group homomorphisms from I to M it follows that I acts trivially on $\phi^{-1}(\overline{y})$.

Let $\mathcal{Y}_n$ be a subcover corresponing to a finite quotient M_n of M. Now let $z \in \mathcal{Y}_n$ such that $\phi(z) = y$. By the above, I acts trivially on $k(z)$. We claim that $\mathcal{O}_z^I$ is étale over $\mathcal{O}_y^I$. To check this is is enough to pass to the completion. But then $\widehat{\mathcal{O}_z} \cong \widehat{\mathcal{O}_y} \otimes_{W(k(y))} W(k(z))$, and I acts trivially on $W(k(z))$. It follows that $\widehat{\mathcal{O}_z}^I = \widehat{\mathcal{O}_y}^I \otimes_{W(k(y))} W(k(z))$ and the claim follows.

Finally, it follows that $\mathcal{Y}_n/G$ is finite etale over $\mathcal{Y}/G \cong \mathcal{X}$. Thus the inverse limit of the $\mathcal{Y}_n/G$ give an extension of $\mathcal{L}$ to $\mathcal{X}$ as desired.

□

3 Proof of Theorem 2

For some positive integer N divisible by ℓ, we have that S spreads out to $\mathcal{S}$ with compactification $\mathcal{S} \subset \mathcal{S}'$ over $\mathcal{O}_E[\frac{1}{N}]$. We let T be the cover of S corresponding to the level where the representation defined by ${}_{\text{et}}\mathbb{V}_\ell$ is trivial mod ℓ^2, and we assume T spreads out to a smooth model $\mathcal{T}$ over $\mathcal{O}_E[\frac{1}{N}]$ which is a Galois cover of $\mathcal{S}$, and let $\pi : \mathcal{T} \to \mathcal{S}$ be the natural map. Let $Q_{CM} \in T(\overline{\mathbb{Q}})$ be a CM point. By increasing N, we assume further that $\mathcal{T} \subset \mathcal{T}'$ is a log-smooth compactification, that the Zariski-closure of Q_{CM} in $\mathcal{T}'$ is contained in $\mathcal{T}$, and that the field of definition of Q_{CM} is unramified away from N. Note that by our assumption, the monodromy elements around all the boundary divisors are unipotent.

The following lemma is essentially the same argument as [7, Lemma 4.4]:

Lemma 3.1. *For all finite places $v \nmid N$ of E, we have $S_v^{\ell-pun} = \mathcal{S}(\overline{\mathcal{O}_{E_v}})$.*

Proof. Let $\mathcal{L} := \pi_{\text{et}}^* V_\ell$ be the corresponding ℓ-adic local system on $\mathcal{T}$. We claim that $\mathcal{L}$ must have trivial monodromy around the special fiber of $\mathcal{T}$, which follows by Lemma 2.6 since Q_{CM} has everywhere potentially good reduction. Thus, $\mathcal{L}$ must extend to $\mathcal{T}$. We let $K = E_v$.

Note that $\pi^*\mathcal{S}(\overline{\mathcal{O}_{E_v}}) = \mathcal{T}(\overline{\mathcal{O}_{E_v}})$ and so it suffices to prove that $T_v^{\ell-pun} = \mathcal{T}(\overline{\mathcal{O}_{E_v}})$ where $T_v^{\ell-pun}$ is defined relative to $\mathcal{L}$. So suppose that $Q \in T_v^{\ell-pun}(\overline{E_v})\backslash\mathcal{T}(\overline{\mathcal{O}_{E_v}})$ and let $z \in \mathcal{T}'(\overline{\mathbb{F}}_v)$ be the reduction of Q at the special fiber. Finally, we denote by $\mathcal{T}'_{un}$ the base change of $\mathcal{T}'$ to the maximal unramified extension K_{un} of K.

Let $x_1, \ldots, x_n \in R := \widehat{\mathcal{O}}_{\mathcal{T}',z}$ be a regular sequence cutting out the irreducible components of the boundary divisors of $\mathcal{T}'\backslash\mathcal{T}$ at Q, and let $\sigma_1, \ldots, \sigma_n$ be the generators of the natural $\mathbb{Z}_\ell(1)^n$ quotient of $\pi_1(R[\frac{1}{x_1}, \ldots, \frac{1}{x_n}])$ corresponding to the complex loops around those same boundary divisors with respect to an identification $\overline{K} \cong \mathbb{C}$.

Identifying the maximal ℓ-power quotient of $\pi_1(\mathcal{O}_{K_{un}})$ with $\mathbb{Z}_\ell(1)$, the natural map

$$\pi_1\left(R\left[\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right]\right) \to \pi_1(\mathcal{O}_{K_{un}})$$

is naturally identified with

$$(a_1, \ldots, a_n) \to \sum_{i=1}^n a_i v_Q(x_i).$$

Thus, since $Q \in T_v^{\ell-pun}$, we conclude that $\prod_{i=1}^n \sigma_i^{a_i}$ acts trivially on $\mathcal{L}_Q$. However, this contradicts Lemma 2.5, which completes the proof.

□

Since we may simply glue in integral models at finitely many places, by Corollary 2.3 we have reduced ourselves to proving the following:

Proposition 3.2. *Let $v \nmid \ell$ be a finite place of E. There exists an integral model $\mathcal{S}$ of S_v over $\mathcal{O}_{E_v}$ such that $S_v^{\ell-pun} = \mathcal{S}(\overline{\mathcal{O}_{E_v}})$. Morover, if $\mathcal{L}$ has torsion-free monodromy, we may choose $\mathcal{S}$ so that $\mathcal{L}$ extends.*

We first reduce to the case where the monodromy of ${}_{\text{et}}\mathbb{V}_\ell$ is trivial mod ℓ (and hence pro-ℓ, torsion free and prime-to-p). To that end, let $f : W \to S$ denote the ℓ^2-level cover of S.

Proposition 3.3. *Proposition 3.2 for W implies Proposition 3.2 for S.*

Proof. Assume that Proposition 3.2 is true for W with the model $\mathcal{W}$. By Lemma 2.1 we may assume $\mathcal{W}$ is projective. Let G be the Galois group of f, and consider the map $h : W \to \mathcal{W}^G$ given by $h(w)_g = g \circ f(w)$. Finally, let $\mathcal{W}_1$ denote the normalization of the closure of $h(W)$. Then $\mathcal{W}_1$ is also an integral model of W, and from construction $W_v^{\ell-pun} = \mathcal{W}_1(\overline{\mathcal{O}_{E_v}})$. Moreover there is a natural group action of G on $\mathcal{W}_1$, and the quotient $\mathcal{S}_1$ is an integral model for S.

Finally, we claim that $\mathcal{L}$ extends to $\mathcal{S}$. Since $\mathcal{S}$ is normal it is enough to prove this locally. But the image of inertia around every point must be simultaneously torsion and yet contained in the monodromy image of $\mathcal{L}$, which is torsion-free. Hence the image of inertia is trivial, which means that $\mathcal{L}$ extends. ☐

Henceforth we take $K = E_v$, $k = \mathbb{F}_v$, and suppress the subscript v in S_v. By Lemma 2.2 we may pass to the ℓ^2-level cover, and therefore assume $\mathcal{L} :=_{\text{et}} \mathbb{V}_\ell$ has unipotent local monodromy. Let $\overline{S}$ denote a log-smooth compactification of S over K. To streamline the proof we introduce the following terminology:

Definition 3.4. Let $\phi : T \to \overline{S}$ be a proper $\overline{S}$-scheme, and let $\mathcal{T}$ be a model of T over $\mathcal{O}_K$. We say that $\mathcal{T}$ is *discerning* if there is a *distinguished* subset $Z \subset \mathcal{T}(\overline{k})$ such that

1. for all $x \in \mathcal{T}(\overline{\mathcal{O}_K})$ such that $\phi(x_{\overline{K}}) \in S(\overline{K})$, the local system $\phi^*_{x_{\overline{K}}}\mathcal{L}$ is potentially unramified iff $x_{\overline{k}} \notin Z$.

2. Z contains the $\overline{k}$-points of the closure of $\phi^{-1}(\overline{S}\backslash S)$.

Note that if $\mathcal{T}$ is discerning then any $\mathcal{T}$-scheme is discerning as well. Also, since every $\overline{k}$-point of $\mathcal{T}$ lifts to a $\overline{\mathcal{O}_K}$-point of $\mathcal{T}$ whose $\overline{K}$-point is contained in S, Z is uniquely determined.

Proposition 3.5. *Let $\phi : X \to \overline{S}$ be a proper $\overline{S}$-scheme which is smooth over K, with a semistable compactification $\mathcal{X}'$ over $\mathcal{O}_K$.*

1. *If $\mathcal{X}'$ itself is discerning, then the distinguished set $Z \subset \mathcal{X}'(\overline{k})$ is closed.*

2. *There exists an admissible modification of $\mathcal{X}'$ which is discerning. Moreover, if the monodromy-image of $\mathcal{L}$ is torsion-free, we may pick this modification such that $\mathcal{L}$ extends to the complement of $\phi^{-1}(\overline{S}\backslash S) \cup \overline{Z}$ in $\mathcal{X}'$.*

Proof. Let $W \subset \mathcal{X}'$ be a (locally closed) boundary stratum contained in the special fiber. Let $F_1, \ldots, F_n$ be the irreducible boundary divisors not contained in the special fiber which contain W, $G_1, \ldots, G_m$ the irreducible boundary divisors in the special fiber containing W, and H the union of the boundary divisors not containing W. Let $z \in W$ be the generic point. Finally, let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be a subset of the regular sequence of generators of $R = \widehat{\mathcal{O}}_{\mathcal{X}',z}$ cutting out $F_1, \ldots, F_m, G_1, \ldots, G_n$.

Let $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n$ be the generators of the natural $\mathbb{Z}_\ell(1)^{m+n}$ quotient of $\pi_1(R[\frac{1}{x_1}, \ldots, \frac{1}{x_m}, \frac{1}{y_1}, \ldots, \frac{1}{y_n}])$ corresponding to the complex loops around the boundary divisors. Let $Q \in X(\overline{K})$ be a point mapping to S and extending to a point in $\mathcal{X}'(\overline{\mathcal{O}_K})$ whose reduction lands in W. By Lemma 2.6, we have that $\prod_{i=1}^m \sigma_i^{v_Q(x_i)} \cdot \prod_{j=1}^n \tau_j^{v_Q(y_j)}$ acts trivially on ${}_{\text{et}}\mathbb{V}_{\ell,z}$ iff $Q \in S^{\ell-pun}$.

Let $\phi : \pi_1(R[\frac{1}{x_1}, \ldots, \frac{1}{x_m}, \frac{1}{y_1}, \ldots, \frac{1}{y_n}]) \to \mathrm{GL}(\mathcal{L}_z)$ denote the monodromy map. Since the image of ϕ is 1 mod ℓ and abelian, we may take logarithms and conclude that $Q \in S^{\ell-pun}$ iff

$$\sum_{i=1}^{m} v_Q(x_i) \log \phi(\sigma_i) + \sum_{j=1}^{n} v_Q(y_j) \log \phi(\tau_j) = 0.$$

For part 1, note that for $z \in W(\overline{k})$ we may pick a lift Q with any positive integral values of the $v_Q(x_i), v_Q(y_j)$. Therefore to be a discerning model, it must be true that either all the $\log \phi(\sigma_i)$ are 0, or that there are no linear relations between them with $\mathbb{Q}_{>0}$-coefficients. Moreover, this must be true for every stratum W. If this is satisfied, Z is simply the union of the $G_i(\overline{k})$ for which the corresponding element $\log \phi(\sigma_i)$ vanishes, and is therefore closed.

We now prove part 2. What follows is a version of [3, Proposition 3.5]. Consider the set P of non-negative rational solutions $(a, b) \in \mathbb{Q}_{\geqslant 0}^{m+n}$ to

$$\sum_{i=1}^{m} a_i \log \phi(\sigma_i) + \sum_{j=1}^{n} b_j \log \phi(\tau_j) = 0$$

According to Lemma 2.5, the intersection of P with the sub-cone $\mathbb{Q}_{\geqslant 0}^{m} \times 0$ corresponding to $F_1, \ldots, F_m$ is 0. We may therefore find an integral subdivision $\mathcal{F}$ of the standard fan on $\mathbb{R}_{\geqslant 0}^{m+n}$ for which P is a union of cones and which does not refine the standard fan on the sub-cone $\mathbb{R}_{\geqslant 0}^{m} \times 0$. If $\overline{W}$ is the closure of W, then this yields a $\mathcal{X}' \backslash \overline{W}$ admissible monomial modification $\phi_W : \mathcal{X}''_W \to \mathcal{X}'$. For the point $Q \in X(\overline{K})$ specializing to W as before, we see that $Q \in S^{\ell-pun}$ iff its specialization in the special fiber of $\mathcal{X}''_W$ is contained in at least one divisor corresponding to a one-dimensional cone of $\mathcal{F}$ not contained in P. Let $Z_W \subset \mathcal{X}''_W(\overline{k})$ be the set of $\overline{k}$-points of the special fiber of the union of these divisors, and note that the special fiber of the strict transform of each F_j is contained in Z_W.

Let $\mathcal{X}'' \to \mathcal{X}'$ be an admissible modification with $\mathcal{X}''$ normal which factors as $\mathcal{X}'' \xrightarrow{\pi_W} \mathcal{X}''_W \to \mathcal{X}'$ for each W. Then it follows that $\mathcal{X}''$ is discerning with distinguished set $Z := \bigcup_W \pi_W^{-1}(Z_W)$.

It remains to show that $\mathcal{L}$ extends if the monodromy is torsion-free. Since $\mathcal{X}''$ is normal, it suffices to show this locally. Note that each fan $\mathcal{F}$ may be chosen to be simplicial, so that $\mathcal{X}''_W$ is semistable, in which case it is clear from the construction that $\mathcal{L}$ extends to $\phi_W^{-1}(W)$.

□

Corollary 3.6. *If $\mathcal{T}$ is discerning, the distinguished set $Z \subset \mathcal{T}(\overline{k})$ is closed.*

Proof. By [2, Thm 4.5], there is a proper morphism $f : X \to \overline{S}$ from a smooth proper scheme X/K which admits a semistable compactification $\mathcal{X}'$. By 3.5, the set $f^{-1}(Z)$ is closed. Since f is proper it follows that Z is closed. □

Given a discerning model $\mathcal{T}$ of $\phi : T \to \overline{S}$, we define $\mathcal{T}^\circ$ to be the complement of $\phi^{-1}(\overline{S} \backslash S) \cup \overline{Z}$.

Proposition 3.7. *There exist finitely many discerning partial compactifications $\mathcal{S}_1, \ldots, \mathcal{S}_m$ of $\overline{S}$ such that every $\overline{K}$-point of $\overline{S}$ extends to an $\overline{\mathcal{O}_K}$-point of $\mathcal{S}_i$ for at least some i. Moreover, if the monodromy-image of $\mathcal{L}$ is torsion-free, we may pick the $\mathcal{S}_i$ such that $\mathcal{L}$ extends to $\mathcal{S}_i^\circ$.*

Proof.

Step 1. By [2, Thm 4.5], for any finite set of points $F \subset \overline{S}(\overline{K})$ there is a proper morphism $f : X \to \overline{S}$ from a smooth proper scheme X/K which is finite étale over a neighborhood of F such that X admits a semistable compactification $\mathcal{X}'_i$ over $\overline{\mathcal{O}_K}$. It follows by Noetherian induction that there is a finite set $\{X_i\}_{i \in I}$ of such $\overline{S}$-schemes such that $f_i : X_i \to \overline{S}$ is finite étale over an open $V_i \subset \overline{S}$ and $\{V_i\}_{i \in I}$ is an open cover of $\overline{S}$.

Step 2. For each $i \in I$, let $\mathcal{X}'_{i,0}$ denote a discerning admissible modification of $\mathcal{X}'_i$, using Proposition 3.5. By Lemma 2.2, for each i there is a $f_i^{-1}(V_i)$-admissible modification $\mathcal{X}''_i \to \mathcal{X}'_{i,0}$, a projective compactification $\mathcal{Y}_i$ of V_i, and a quotient $\mathcal{X}''_i \to \mathcal{Y}_i$ by a finite group G_i which is étale on $f_i^{-1}(V_i)$. The $\mathcal{X}''_i$ are also discerning. After taking normalizations we may assume $\mathcal{X}''_i$ and $\mathcal{Y}_i$ are normal. Since the morphism $X''_i \to \overline{S}$ factors through $X''_i \to Y_i$ on the level of function fields, it follows it factors through a V_i-admissible modification $\pi_i : Y_i \to \overline{S}$. Thus, $\mathcal{Y}_i$ is discerning. Let $D_i \subset \overline{S}$ be the open subset over which π_i is not an isomorphism; since $V_i \subset \overline{S} \backslash D_i$, we have $\bigcap_{i \in I} D_i = \varnothing$.

Step 3. Let $U_i \subset \bar{S}^{\text{an}}$ be neighborhoods of the D_i^{an} which have no intersection. By Lemma 2.4, there is an admissble modification $\mathcal{Y}'_i \to \mathcal{Y}_i$ such that, setting $\pi'_i : Y'_i \to \overline{S}$ to be the natural map, the complement of $\pi_i'^{-1}(U_i)$ is contained in the rigid fiber of $\mathcal{Y}'_i \backslash \overline{\pi_i'^{-1}(D_i)}$. Let $\mathcal{X}_i^\dagger \to \mathcal{Y}'_i$ be the normalization of the reduction of the base-change of $\mathcal{X}''_i \to \mathcal{Y}_i$. Then $\mathcal{Y}'_i$ is the quotient of $\mathcal{X}_i^\dagger$ by G_i since both are normal. We set $\mathcal{S}_i := \overline{S} \cup_{\overline{S} \backslash D_i} \mathcal{Y}'_i \backslash \overline{\pi_i'^{-1}(D_i)}$. By construction, every $\mathcal{S}_i$ is discerning, since every integral point of $\mathcal{S}_i$ is an integral point of $\mathcal{Y}_i'\backslash\overline{\pi_i'^{-1}(D_i)}$. Moreover, every $\overline{K}$-point of $\overline{S}$ is contained in the complement of one of the U_i, hence extends to an $\overline{\mathcal{O}_K}$ point of $\mathcal{S}_i$.

Step 4. Finally, if the monodromy is torsion-free, then by Proposition 3.5 we may pick $\mathcal{X}_{i,0}'$ such that $\mathcal{L}$ extends to $\left(\mathcal{X}_{i,0}'\right)^\circ$ and therefore $\left(\mathcal{X}_i''\right)^\circ$ and $(\mathcal{X}_i^\dagger)^\circ$, in which case it'll extend to each of the $(\mathcal{Y}_i')^\circ\backslash\overline{\pi_i'^{-1}(D_i)}$ by Proposition 2.7, and finally to $\mathcal{S}_i^\circ$ by construction. This completes the proof.

□

We now complete the proof of Proposition 3.2. Let $Z_i \subset \mathcal{S}_i$ denote the closed subschemes as in Definition 3.4. By Nagata's compactification theorem we may embed each $\mathcal{S}_i$ in a proper $\mathcal{S}_i'$, which is therefore a proper model of $\overline{S}$.

Let $\mathcal{S}_0$ be a normal model of $\overline{S}$ with admissible modifications $g_i : \mathcal{S}_0 \to \mathcal{S}_i'$ for each i. We claim that $\mathcal{S}_0$ is a discerning. Define $Z_0 \subset \mathcal{S}_0$ by $Z_0 := \bigcup_{i\in I} g_i^{-1}(Z_i)$. Clearly Z_0 contains $\overline{S}\backslash S$. A $\bar{K}$-point x of S extends to an integral point of $\mathcal{S}_0$, and the extension specializes to Z_0 if and only if x extends and in some $\mathcal{S}_i$ and specializes to Z_i, which is the case if and only if it is not potentially unramified.

By Definition 3.4 and Corollary 3.6, the set $\overline{Z_0}\bigcup(\overline{S}\backslash S)$ is closed. By taking $\mathcal{S}$ to be the complement of $\overline{Z_0}\bigcup(\overline{S}\backslash S)$ in $\mathcal{S}_0$, we obtain the desired model of S.

Finally, observe by construction and Proposition 3.7 that if the monodromy is torsion-free, then $\mathcal{L}$ extends locally across every point of $\mathcal{S}$. Since $\mathcal{S}$ is normal this extension is canonical, and hence $\mathcal{L}$ extends to all of $\mathcal{S}$. This completes the proof of Proposition 3.2, and thus Theorem 2. □

4 Proof of Theorem 1

By [6, §12] all CM points lie in $S_v^{\ell-pun}$ for all $v \nmid \ell$. Let ℓ_1 be an odd rational prime inert in E. Let $\mathcal{S}_1, \mathcal{S}_2$ be the integral models in Theorem 2 corresponding to $\ell = \ell_1, \ell_2$ respectively. We may simply form $\mathcal{S}$ by gluing $\mathcal{S}_1 \times_{\mathcal{O}_E} \mathcal{O}_E[\frac{1}{\ell_1}]$ with $\mathcal{S}_2 \times_{\mathcal{O}_E} \mathcal{O}_{E,(\ell_1)}$ along S, and the theorem is proved. □

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