A HODGE-THEORETIC PROOF OF HWANG'S THEOREM ON BASE MANIFOLDS OF LAGRANGIAN FIBRATIONS

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ABSTRACT. We give a Hodge-theoretic proof of Hwang's theorem, which says that if the base of a Lagrangian fibration of an irreducible holomorphic symplectic manifold is smooth, it must be projective space.

1. INTRODUCTION

Let X be an irreducible holomorphic symplectic manifold of dimension $2n$, and let $f : X \to B$ be a Lagrangian fibration over a base of dimension n. In this note, we reprove the following well-known result of Hwang [Hwa08].

Theorem 1.1 (Hwang). *Let $f : X \to B$ be a Lagrangian fibration of an irreducible holomorphic symplectic manifold of dimension $2n$. If B is smooth, then $B \cong \mathbb{P}^n$.*

The proof that we present below is inspired by Hwang's original argument, but perhaps clarified by the consistent use of Hodge theory, especially the theory of variations of Hodge structure. There are two other important inputs. The first is the following enhancement of Mori's characterization of projective space, due to Cho, Miyaoka, and Shepherd-Barron [CMSB02] and Kebekus [Keb02].

Theorem 1.2 (Cho–Miyaoka–Shepherd-Barron, Kebekus). *Let Y be a uniruled smooth projective variety and suppose there is a dense Zariski open subset $Y^\circ \subseteq Y$ such that every nonconstant rational curve $g : C \to Y$ meeting Y° satisfies $-\deg g^* \omega_Y \geq \dim Y + 1$. Then Y is isomorphic to projective space.*

The second input is a result of Voisin [Voi18] about the variation of Hodge structure on the first cohomology of the fibers of a Lagrangian fibration. Let $B^\circ \subseteq B$ be the open set over which the Lagrangian fibration f is smooth, and $f^\circ : X^\circ \to B^\circ$ the base change of f to B°. It is known that the discriminant locus $D = B \setminus B^\circ$ is a nonempty divisor. We denote by $V = R^1 f^\circ_* \mathbb{Q}_{X^\circ}$ the polarized variation of Hodge structure of weight 1 on B°; by $\mathcal{V}$ the underlying flat vector bundle; by ∇ the Gauss-Manin connection on $\mathcal{V}$; and by $F^1 \mathcal{V} = f^\circ_* \Omega^1_{X^\circ/B^\circ}$ the Hodge bundle of type $(1,0)$.

Theorem 1.3 (Voisin). *$V \otimes \mathbb{R}$ is irreducible as a real variation of Hodge structure.*

The anticanonical degree of a curve (as well as its deformation theory) is linked to the variation V since contraction with the symplectic form yields a natural isomorphism $\mathscr{T}_{B^\circ} \cong F^1 \mathcal{V}$. Work of Matsushita [Mat05] extends this to an isomorphism between $\mathscr{T}_B$ and the "canonical extension" of $F^1 \mathcal{V}$ in codimension one, and this will be an important ingredient in our approach.

It is known by [Mat99] that B is a Fano manifold (and therefore uniruled). The strategy is to assume the condition in Theorem 1.2 is not satisfied, so that there is a low degree rational curve meeting B°, and conclude from this that: (1) the curve deforms to cover B; and (2) the variation V must have a nontrivial fixed part when restricted to C. We then obtain a nontrivial splitting of V on the universal family of curves and thus on a finite cover of B. To finish, we show that this splitting descends to B°, where it of

course contradicts Theorem 1.3. In our approach, the first step is proved using a Hodge module version of Matsushita's result combined with a straightforward functoriality property of Hodge modules to show that $g^* \mathscr{T}_B$ is semi-positive (see Proposition 2.2) and that its maximal trivial quotient comes from the fixed part of the variation (see Proposition 2.4). As the low degree curve which is assumed to exist may a priori not deform, it can intersect the discriminant divisor in its singular locus, and the Hodge module perspective provides a version of Matsushita's theorem that works in higher codimension. The descent step is straightforward in the case that f is non-isotrivial (using the result of [Bak22], see Remark 6.3), but we do not actually use this. Instead, we prove the descent in general using a rather tricky argument which is a reinterpretation of an argument from Hwang's paper using Hodge theory to simplify the steps. Note that the descent step must in some sense be subtle—in fact, it is the most difficult part of our proof—as it is easy to construct examples where V splits off a factor on a finite cover that is constant on a covering family of rational curves (see Remark 7.1).

Remark 1.4. The recent preprint by Li and Tosatti [LT23] contains another proof of Hwang's theorem that follows the same overall strategy, but has a more analytic flavor. Instead of Hodge theory, Li and Tosatti rely on the theory of special Kähler manifolds [Fre99]. They use the curvature of the special Kähler metric on B° to show that the restriction of $\mathscr{T}_B$ to a rational curve of minimal degree is semipositive, and that any trivial quotient of $g^* \mathscr{T}_B$ actually comes from a subbundle that is flat with respect to the Chern connection of the metric. For the descent step at the end of the proof, however, they depend on one of the results from our paper.

Outline. In Section 2 we prove the results on the semipostivity of $g^* \mathscr{T}_B$ and the relation between the maximal trivial quotient and the fixed part of the variation. We give two proofs: the first in the projective case is more geometric, using results of Kollár; the second as a consequence of the functoriality of Hodge module extensions. In Section 3 we make some general remarks relating subvariations to foliations on B. In Section 4 we use the existence of a low degree curve to construct a nontrivial splitting of the variation on a finite cover. In Section 5, we study the geometry of the subvarieties swept out by connected components of the family of rational curves through a given point. (For the sake of exposition, we reprove all the necessary results, which are due to Araujo [Ara06] and Hwang [Hwa07], even though this makes the paper longer.) In Section 6, we describe the main ingredient for descending the splitting to B, using ideas from [Hwa08]. Finally, we prove Theorem 1.1 in Section 7.

Notation and terminology. By a *rational curve* $g : C \to B$ we always mean a nonconstant morphism from $C \cong \mathbb{P}^1$. If the image meets the open subset B°, we denote by $g^\circ : C^\circ := g^{-1}(B^\circ) \to B^\circ$ the restriction of g, and in general the superscript "$\circ$" always stands for objects that are related to B°. We denote by $V_C := (g^\circ)^* V$ the pullback of the variation, and likewise for any pullback. We typically use Roman letters V, W to denote local systems and script letters $\mathcal{V}, \mathcal{W}$ to denote the associated flat vector bundles.

2. Restriction of the tangent bundle to rational curves

In this section, we use results from Hodge theory to analyze the splitting behavior of the tangent bundle $\mathscr{T}_B$ on rational curves in B that meet the smooth locus B° of the Lagrangian fibration $f : X \to B$. Here it will be more convenient to work with the variation of Hodge structure on $R^{2n-1} f_*^\circ \mathbb{Q}_{X^\circ}$ instead of the one on $R^1 f_*^\circ \mathbb{Q}_{X^\circ}$. This makes no essential difference, because the Hard Lefschetz theorem says that, after

tensoring by $\mathbb{R}$, the two variations are isomorphic up to a Tate twist by $\mathbb{R}(-n+1)$; the isomorphism is induced by cup product with the $(n-1)$-th power of a Kähler form on X. We will therefore change the notation slightly, and—in this section only—use the symbol V for the variation of Hodge structure on $R^{2n-1}f_*^{\circ}\mathbb{Q}_{X^{\circ}}$. The relevant Hodge bundle is then $F^n\mathcal{V} = R^{n-1}f_*^{\circ}\omega_{X^{\circ}/B^{\circ}}$; it is of course isomorphic to $f_*^{\circ}\Omega^1_{X^{\circ}/B^{\circ}}$.

Our starting point is the following result by Matsushita [Mat05], which describes the tangent bundle of B in terms of Hodge theory.

Theorem 2.1. *There is an isomorphism $R^{n-1}f_*\omega_{X/B} \cong \mathscr{T}_B$, whose restriction to B° is the natural isomorphism induced by the holomorphic symplectic form.*

Proof. Matsushita proves that the natural isomorphism on B° extends to $R^{n-1}f_*\omega_X \cong \Omega^{n-1}_B \cong \omega_B \otimes \mathscr{T}_B$. Tensoring both sides by ω_B^{-1} then gives the desired result. Matsushita only states the theorem for projective X, but his proof carries over to the case where X is a compact Kähler manifold. For an alternative proof, see [Sch23, §4]. $\qquad\square$

Let $g\colon C \to B$ be a nonconstant morphism from $C \cong \mathbb{P}^1$ whose image intersects B°. Like any vector bundle on $\mathbb{P}^1$, the pullback $g^*\mathscr{T}_B$ decomposes as

$$g^*\mathscr{T}_B \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n) \tag{1}$$

for certain integers $a_1,\ldots,a_n$. Our first result is that $g^*\mathscr{T}_B$ is semi-positive.

Proposition 2.2. *The pullback $g^*\mathscr{T}_B$ is semi-positive, in the sense that $a_1,\ldots,a_n \geq 0$.*

We will first present a geometric proof in the case when X is projective; afterwards, we will give a Hodge-theoretic proof that also works when X is a compact Kähler manifold. Let us denote by $h\colon Y \to C$ the base change of the Lagrangian fibration, and let $r\colon \tilde{Y} \to Y$ be a resolution of singularities of Y that is an isomorphism over the preimage of $C^{\circ} = g^{-1}(B^{\circ})$. For dimension reasons, the composition $\tilde{h} = r \circ h\colon \tilde{Y} \to C$ is still flat of relative dimension n. We get a commutative diagram

$$
\begin{array}{ccccc}
\tilde{Y} & \xrightarrow{\ r\ } & Y & \longrightarrow & X \\
 & \diagdown{\scriptstyle\tilde{h}} & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\
 & & C & \xrightarrow{\ g\ } & B
\end{array}
$$

in which the square is Cartesian. The key step in the proof is the following lemma.

Lemma 2.3. *There is a morphism of sheaves*

$$R^{n-1}\tilde{h}_*\omega_{\tilde{Y}/C} \to g^*\mathscr{T}_B$$

that restricts to the natural isomorphism over the open subset $C^{\circ} = g^{-1}(B^{\circ})$.

Proof. Before giving the proof, we need to review a few facts about relative dualizing sheaves and base change. Here we probably have to assume that X is projective (although it is known by [Cam21] that Lagrangian fibrations are projective locally in the analytic topology on the base). In this algebraic setting, a good reference for duality theory is [Sta18, Tag 0DWE]. To begin with, B is smooth and f is flat, and so all fibers of f are local complete intersections in X and therefore Gorenstein. Since f is proper and flat, it follows that the relative dualizing sheaf $\omega_{X/B}$ is isomorphic to $\omega_X \otimes f^*\omega_B^{-1}$, and that the relative dualizing complex is $\omega_{X/B}[n]$. Moreover, in this situation, the relative dualizing sheaf commutes with arbitrary base change.

Now we begin the proof. Flatness of f and the base change property for the relative dualizing sheaf give us an isomorphism

$$g^*\mathbf{R}f_*\omega_{X/B} \cong \mathbf{R}h_*\omega_{Y/C}.$$

Since all the higher direct images $R^i f_* \omega_{X/B}$ are locally free (by Matsushita's theorem), we get in particular that

$$g^* \mathscr{T}_B \cong g^* R^{n-1} f_* \omega_{X/B} \cong R^{n-1} h_* \omega_{Y/C}. \qquad (2)$$

Consider now the resolution of singularities $r \colon \tilde{Y} \to Y$. The trace map gives us a morphism $r_* \omega_{\tilde{Y}} \to \omega_Y$ that is an isomorphism over the smooth locus of Y and therefore injective. The induced morphism

$$R^{n-1} \tilde{h}_* \omega_{\tilde{Y}/C} \to R^{n-1} h_* \omega_{Y/C} \qquad (3)$$

is of course still an isomorphism over C°. We then get the desired morphism by composing (3) and (2); it is clearly the natural isomorphism over C°. $\qquad \square$

We can now prove that $g^* \mathscr{T}_B$ is always semi-positive.

Proof of Proposition 2.2. From Lemma 2.3, we get a short exact sequence

$$0 \to R^{n-1} \tilde{h}_* \omega_{\tilde{Y}/C} \to g^* \mathscr{T}_B \to \mathscr{Q} \to 0,$$

in which $\mathscr{Q}$ is supported on the finite set $C \setminus C^\circ$ and therefore a torsion sheaf. This reduces the problem to the semi-positivity of $R^{n-1} \tilde{h}_* \omega_{\tilde{Y}/C}$. Since $\tilde{h} \colon \tilde{Y} \to C$ is a Kähler fiber space over a curve, this is the content of a classical result by Fujita [Fuj78]. Alternatively, one can argue using Kollár's results [Kol86a] about higher direct images of dualizing sheaves. Since $C \cong \mathbb{P}^1$, we have

$$R^i h_* \omega_{\tilde{Y}/C} \otimes \mathscr{O}(-1) \cong R^i h_* \omega_{\tilde{Y}} \otimes \mathscr{O}(1),$$

and this bundle has no higher cohomology by Kollár's vanishing theorem (which applies to any morphism from a compact Kähler manifold to a projective variety). But on $\mathbb{P}^1$, this is only possible if $R^i h_* \omega_{\tilde{Y}/C}$ is semi-positive. $\qquad \square$

We also need to understand the trivial part in the decomposition (1), at least in the case of generic rational curves. Let us therefore assume from now on that $g \colon C \to B$ is an immersed rational curve that intersects the discriminant divisor $D = B \setminus B^\circ$ transversely. The preimage $Y = X \times_C B$ is then already smooth, and (2) gives us an isomorphism

$$g^* \mathscr{T}_B \cong R^{n-1} h_* \omega_{Y/C}.$$

Suppose that in the decomposition (1), we have $a_1, \ldots, a_r = 0$ and $a_{r+1}, \ldots, a_n > 0$. In that case, the bundle $g^* \mathscr{T}_B$ has a canonical trivial quotient

$$g^* \mathscr{T}_B \to \mathscr{O}_C^{\oplus r}.$$

On the open subset $C^\circ = g^{-1}(B^\circ)$, we have a polarized variation of Hodge structure of weight $2n - 1$, whose underlying local system is $R^{2n-1} h_*^\circ \mathbb{Q}_{Y^\circ}$; here $h^\circ \colon Y^\circ \to C^\circ$ is the restriction of h. Our next result identifies the maximal trivial quotient of $g^* \mathscr{T}_B$ as coming from the fixed part of the variation of Hodge structure on $R^{2n-1} h_*^\circ \mathbb{Q}_{Y^\circ}$.

Proposition 2.4. *The fixed part of the variation of Hodge structure on $R^{2n-1} h_*^\circ \mathbb{Q}_{Y^\circ}$ has rank exactly $2r$, and the Hodge bundle of type $(n, n-1)$ of the fixed part projects isomorphically to the maximal trivial quotient of $g^* \mathscr{T}_B$.*

Proof. To simplify the notation, let us set $V_C = R^{2n-1} h_*^\circ \mathbb{Q}_{Y^\circ}$, and let us denote by $\mathcal{V}_C$ the underlying flat vector bundle, and by $F^n \mathcal{V}_C$ the Hodge bundle of type $(n, n-1)$. Since h has relative dimension n, the restriction of $R^{n-1} h_* \omega_{Y/C}$ to the open subset C° is canonically isomorphic to $F^n \mathcal{V}_C$. In fact, more is true. Let $\tilde{\mathcal{V}}_C^{>-1}$ denote Deligne's canonical extension of the flat vector bundle $\mathcal{V}_C$; it is uniquely characterized by the condition that the connection on $\mathcal{V}_C$ has logarithmic poles on $\tilde{\mathcal{V}}_C^{>-1}$ and that the

residues of the connection at each point of $C \setminus C^\circ$ have eigenvalues contained in the interval $(-1, 0]$. With this notation, Kollár [Kol86a] proves that

$$R^{n-1} h_* \omega_{Y/C} \cong \tilde{\mathcal{V}}_C^{>-1} \cap j_*(F^n \mathcal{V}_C), \tag{4}$$

where $j \colon C^\circ \to C$ is the inclusion.

The fixed part of V_C is a $\mathbb{Q}$-Hodge structure of type $(n, n-1) + (n-1, n)$, and so its dimension is an even number, say $2s$. The fixed part then contributes a trivial summand of rank $2s$ to the canonical extension $\tilde{\mathcal{V}}_C^{>-1}$, and, due to (4), a trivial summand of rank s to the bundle $R^{n-1} h_* \omega_{Y/C}$. This gives $r \geq s$. To prove that $r = s$, we need to argue that any nontrivial morphism of the form

$$R^{n-1} h_* \omega_{Y/C} \to \mathscr{O}_C \tag{5}$$

must come from the fixed part of V_C. This is a straightforward computation with duality. First, we observe that since C is a curve, the derived pushforward of the line bundle $\omega_{Y/C}$ decomposes as

$$\mathbf{R} h_* \omega_{Y/C} \cong \bigoplus_{i=0}^{n} R^i h_* \omega_{Y/C}[-i]$$

in the derived category of coherent sheaves on C. (This kind of result is due to Kollár [Kol86b] in general, but only needs elementary homological algebra in the case of curves.) Consequently, $R^{n-1} h_* \omega_{Y/C}$ is a direct summand of $\mathbf{R} h_* \omega_{Y/C}[n-1]$.

Grothendieck duality for the morphism $h \colon Y \to C$ yields

$$\operatorname{Hom}_C\Big(\mathbf{R} h_* \omega_{Y/C}[n-1], \mathscr{O}_C\Big) \cong \operatorname{Hom}_Y\Big(\omega_{Y/C}[n-1], \omega_{Y/C}[n]\Big) \cong H^1(Y, \mathscr{O}_Y).$$

After substituting in the above formula for the direct image and remembering that $H^1(C, \mathscr{O}_C) = 0$ (because $C \cong \mathbb{P}^1$), we get an isomorphism

$$\operatorname{Hom}_C\Big(R^{n-1} h_* \omega_{Y/C}, \mathscr{O}_C\Big) \cong H^1(Y, \mathscr{O}_Y).$$

Now Y is a compact Kähler manifold, and so by classical Hodge theory, the morphism $H^1(Y, \mathbb{C}) \to H^1(Y, \mathscr{O}_Y)$ is surjective. After some compatibility checking, it follows that any morphism as in (5) comes about in the following way: at any point $c \in C^\circ$, the restriction of (5) is the linear functional $H^{n,n-1}(Y_c) \to \mathbb{C}$ obtained by integration against the restriction of a fixed class in $H^1(Y, \mathbb{C})$ to the fiber $Y_c = h^{-1}(c)$. Since restriction to the smooth fibers of $h \colon Y \to C$ determines a morphism of variations of Hodge structure

$$H^1(Y, \mathbb{Q}) \otimes R^{2n-1} h_*^\circ \mathbb{Q}_{Y^\circ} \to R^{2n} h_*^\circ \mathbb{Q}_{Y^\circ} \cong \mathbb{Q}_C(-n),$$

we conclude (by the semi-simplicity of polarized variations of Hodge structure) that the fixed part of $R^{2n-1} h_*^\circ \mathbb{Q}_{Y^\circ}$ must have rank at least $2r$. $\qquad \square$

Remark 2.5. It is possible that the construction with relative dualizing sheaves and base change still works in the analytic setting, but we have not been able to find a reference for the necessary technical results.

In the remainder of this chapter, we explain how one can modify the argument above to make it work when X is just a compact Kähler manifold. The proof relies on a few basic properties of Hodge modules [Sai88, Sai90]. Let M denote the polarized Hodge module of weight $\dim B + (2n-1)$ with strict support B, associated to the variation of Hodge structure $V = R^{2n-1} f_* \mathbb{Q}_{X^\circ}$ on B° by [Sai90, Thm. 3.21]. Let $\mathcal{M}$ denote the (regular holonomic) left $\mathscr{D}_B$-module underlying M, and let $F_\bullet \mathcal{M}$ be its Hodge

filtration. By Saito's direct image theorem [Sai88, Thm. 5.3.1] (and [Sai91] for the constant Hodge module on a compact Kähler manifold), we have $F_{-n-1}\mathcal{M} = 0$, and

$$\omega_B \otimes F_{-n}\mathcal{M} \cong R^{n-1}f_*\omega_X.$$

Together with Theorem 2.1, this implies that $F_{-n}\mathcal{M} \cong \mathscr{T}_B$, and so we can also think of the tangent sheaf as being part of a Hodge module on B. We may sometimes refer to the coherent $\mathscr{O}_B$-module $F_{-n}\mathcal{M}$ as the "Hodge module extension" of $F^n\mathcal{V}$.

There is an alternative description of the coherent sheaf $F_{-n}\mathcal{M}$ using resolution of singularities. Let $\mu\colon \tilde{B} \to B$ be a log resolution of the discriminant divisor $D = B \setminus B^\circ$ that is an isomorphism over the open subset over which D is a divisor with normal crossings, hence in particular over B°. Let $\mathcal{V}$ denote the flat vector bundle underlying the variation of Hodge structure $V = R^{2n-1}f_*^\circ\mathbb{Q}_{X^\circ}$, and let $F^n\mathcal{V} = R^n f_*^\circ \omega_{X^\circ/B^\circ}$ be the Hodge bundle of type $(n, n-1)$. As before, let $\tilde{\mathcal{V}}^{>-1}$ denote Deligne's canonical extension of the flat vector bundle $\mathcal{V}$; this is the unique extension of $\mathcal{V}$ to a vector bundle on $\tilde{B}$ on which the connection has logarithmic poles with residues having eigenvalues in the interval $(-1, 0]$. By Schmid's nilpotent orbit theorem, the Hodge bundle $F^n\mathcal{V}$ extends to a subbundle $F^n\tilde{\mathcal{V}}^{>-1} = \tilde{\mathcal{V}}^{>-1} \cap j_*(F^n\mathcal{V})$, where $j\colon B^\circ \to \tilde{B}$ is the inclusion. According to [Sai91, Thm. 2.4 and (2.2.2)], we then have

$$\omega_B \otimes F_{-n}\mathcal{M} \cong \mathbf{R}\mu_*\Big(\omega_{\tilde{B}} \otimes F^n\tilde{\mathcal{V}}^{>-1}\Big). \tag{6}$$

This morphism is an isomorphism over the open subset where $D = B \setminus B^\circ$ is a divisor with normal crossings.

We can now present the alternative proof of Proposition 2.2. Let $g\colon C \to B$ be a rational curve meeting the smooth locus B° of the Lagrangian fibration. Let V_C be the variation of Hodge structure obtained by pulling back $V = R^{2n-1}f_*^\circ\mathbb{Q}_{X^\circ}$ to the open subset $C^\circ = g^{-1}(B^\circ)$, and let M_C be its extension to a polarized Hodge module of weight $\dim C + (2n-1)$ with strict support C. Let $(\mathcal{M}_C, F_\bullet\mathcal{M}_C)$ be the underlying filtered left $\mathscr{D}_C$-module. If we again write $\tilde{\mathcal{V}}_C^{>-1}$ for Deligne's canonical extension, then for the same reason as above, we have

$$F_{-n}\mathcal{M}_C \cong F^n\tilde{\mathcal{V}}_C^{>-1}, \tag{7}$$

Since V_C is a polarized variation of Hodge structure with quasi-unipotent local monodromy on a curve, this vector bundle is known to be semi-positive [Pet84].[1] Alternatively, using the notation from the proof of Proposition 2.2, we can observe that $F^n\tilde{\mathcal{V}}_C^{>-1} \cong R^{n-1}h_*\omega_{\tilde{Y}/C}$, which is semi-positive for the reason explained above. Either way, to relate this semi-positivity to that of $g^*\mathscr{T}_B$, we need the following variant of our earlier Lemma 2.3.

Lemma 2.6. *There is a morphism of sheaves $F_{-n}\mathcal{M}_C \to g^*\big(F_{-n}\mathcal{M}\big)$ that restricts to the natural isomorphism over $C^\circ = g^{-1}(B^\circ)$.*

Proof. This kind of morphism is constructed in [Sch12, §13-14] in much greater generality. We present here a more elementary argument that reduces the input from the theory of Hodge modules to a minimum. Recall from (6) that we have an isomorphism

$$\mathbf{R}\mu_*\Big(\omega_{\tilde{B}} \otimes F^n\tilde{\mathcal{V}}^{>-1}\Big) \cong \omega_B \otimes F_{-n}\mathcal{M}.$$

The right-adjoint of the functor $\mathbf{R}\mu_*$ is the exceptional inverse image functor $\mu^! = \omega_{\tilde{B}} \otimes \mu^*\omega_B^{-1} \otimes \mathbf{L}\mu^*$. After applying this to the above isomorphism, we get a morphism

$$F^n\tilde{\mathcal{V}}^{>-1} \to \mathbf{L}\mu^*\big(F_{-n}\mathcal{M}\big)$$

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[1]Peters claims to be working with the canonical extension for the interval $[0, 1)$, but his signs in the formula for the Hodge norm are wrong, and so he should really be using the interval $(-1, 0]$.

in the derived category; since $F_{-n}\mathcal{M} \cong \mathscr{T}_B$ is locally free, this is a morphism of sheaves

$$F^n \tilde{\mathcal{V}}^{>-1} \to \mu^*(F_{-n}\mathcal{M}). \tag{8}$$

By construction, this morphism is an isomorphism outside the exceptional divisor of μ, and therefore injective.

The rational curve $g\colon C \to B$ intersects the smooth locus B°, and therefore lifts uniquely to a rational curve $\tilde{g}\colon C \to \tilde{B}$, making the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\ \tilde{g}\ } & \tilde{B} \\
& \searrow{\scriptstyle g} & \downarrow{\scriptstyle \mu} \\
& & B
\end{array}
$$

After pulling back (8) along $\tilde{g}$, we obtain a morphism

$$\tilde{g}^*(F^n\tilde{\mathcal{V}}^{>-1}) \to g^*(F_{-n}\mathcal{M}).$$

The properties of the canonical extension give us a natural morphism $\tilde{\mathcal{V}}_C^{>-1} \to \tilde{g}^*(\tilde{\mathcal{V}}^{>-1})$ that is an isomorphism over C°. Consequently, we have a morphism

$$F^n\tilde{\mathcal{V}}_C^{>-1} \to \tilde{g}^*(F^n\tilde{\mathcal{V}}^{>-1}),$$

and after composing the two morphisms and remembering the identity in (7), we get the desired morphism

$$F_{-n}\mathcal{M}_C \to g^*(F_{-n}\mathcal{M}).$$

By construction, it is an isomorphism over the open subset $C^\circ = g^{-1}(B^\circ)$. $\qquad\square$

This gives us an injective morphism from the semi-positive vector bundle $F_{-n}\mathcal{M}_C \cong F^n\tilde{\mathcal{V}}_C^{>-1}$ into $g^*(F_{-n}\mathcal{M}) \cong g^*\mathscr{T}_B$, and since this morphism is an isomorphism over C°, we can argue as before to finish the proof of Proposition 2.2.

Now suppose that the rational curve $g\colon C \to B$ intersects the discriminant divisor D transversely. This means that it stays inside the open subset where $F_{-n}\mathcal{M}$ is described by the canonical extension, and because the intersection is transverse, the morphism

$$F^n\tilde{\mathcal{V}}_C^{>-1} \to g^*\mathscr{T}_B$$

is an isomorphism. Moreover, the fiber product $Y = C \times_B X$ is again smooth, and so by (4), we have $F^n\tilde{\mathcal{V}}_C^{>-1} \cong R^{n-1}h_*\omega_{Y/C}$, where $h\colon Y \to C$ is the base change of the Lagrangian fibration. Proposition 2.4 therefore still applies in this setting.

3. Foliations and variations of Hodge structure

The purpose of this section is to prove a couple of small technical results that relate the variation of Hodge structure on $V = R^1 f_*^\circ \mathbb{Q}_{X^\circ}$ to the theory of foliations. Here $f\colon X \to B$ is again a Lagrangian fibration from a compact Kähler manifold X.

Recall that $\mathcal{V}$ denotes the underlying flat vector bundle, $\nabla\colon \mathcal{V} \to \Omega^1_{B^\circ} \otimes \mathcal{V}$ the Gauss-Manin connection, and $F^1\mathcal{V} = f_*^\circ \Omega^1_{X^\circ/B^\circ}$ the Hodge bundle. We have $F^1\mathcal{V} \cong \mathscr{T}_{B^\circ}$, with the isomorphism induced by contraction with the holomorphic symplectic form $\sigma \in H^0(X, \Omega_X^2)$. We are going to need the following result by Freed [Fre99] that describes the Lie bracket on the tangent bundle in terms of the connection.

Lemma 3.1. *For any two holomorphic vector fields $\xi, \eta \in \mathscr{T}_{B^\circ}$, one has*

$$\nabla_\xi(\eta) - \nabla_\eta(\xi) = [\xi, \eta].$$

Proof. Let $U \subseteq B^\circ$ be an open subset and $\xi, \eta \in H^0(U, \mathscr{T}_B)$ be two holomorphic vector fields. Since $f^\circ \colon X^\circ \to B^\circ$ is a fiber bundle, we can lift them to smooth vector fields $\tilde{\xi}, \tilde{\eta} \in A^0\big(f^{-1}(U), \mathscr{T}_X\big)$; then $[\tilde{\xi}, \tilde{\eta}]$ is a smooth lifting of the Lie bracket $[\xi, \eta]$. The isomorphism $\mathscr{T}_{B^\circ} \cong F^1 \mathcal{V}$ takes the holomorphic vector field ξ to the fiberwise cohomology class of the smooth $(1,0)$-form

$$\Omega_{\tilde{\xi}} = \mathrm{int}(\tilde{\xi})(\sigma) \in A^{1,0}\big(f^{-1}(U)\big),$$

obtained by contracting the vector field $\tilde{\xi}$ against the holomorphic symplectic form. In these terms, the Gauss-Manin connection $\nabla_\xi(\eta) \in H^0(U, \mathcal{V})$ is given by taking the fiberwise cohomology class of the smooth 1-form

$$\mathrm{int}(\tilde{\xi})(d\Omega_{\tilde{\eta}}) \in A^1\big(f^{-1}(U)\big).$$

Now we compute that, for any smooth vector field λ on $f^{-1}(U)$, one has

$$\Big(\mathrm{int}(\tilde{\xi})(d\Omega_{\tilde{\eta}}) - \mathrm{int}(\tilde{\eta})(d\Omega_{\tilde{\xi}}) \Big)(\lambda) = d\Omega_{\tilde{\eta}}(\tilde{\xi}, \lambda) - d\Omega_{\tilde{\xi}}(\tilde{\eta}, \lambda)$$

$$= \Big(\tilde{\xi} \cdot \sigma(\tilde{\eta}, \lambda) - \lambda \cdot \sigma(\tilde{\eta}, \tilde{\xi}) - \sigma\big(\tilde{\eta}, [\tilde{\xi}, \lambda]\big) \Big) - \Big(\tilde{\eta} \cdot \sigma(\tilde{\xi}, \lambda) - \lambda \cdot \sigma(\tilde{\xi}, \tilde{\eta}) - \sigma\big(\tilde{\xi}, [\tilde{\eta}, \lambda]\big) \Big)$$

$$= \sigma\big([\tilde{\xi}, \tilde{\eta}], \lambda\big) + \lambda \cdot \sigma(\tilde{\xi}, \tilde{\eta}),$$

using the coordinate-free formula for the exterior derivative, and the identity

$$0 = (d\sigma)(\lambda, \tilde{\xi}, \tilde{\eta}) = \lambda \cdot \sigma(\tilde{\xi}, \tilde{\eta}) + \tilde{\xi} \cdot \sigma(\tilde{\eta}, \lambda) + \tilde{\eta} \cdot \sigma(\lambda, \tilde{\xi})$$
$$+ \sigma\big(\lambda, [\tilde{\xi}, \tilde{\eta}]\big) + \sigma\big(\tilde{\xi}, [\tilde{\eta}, \lambda]\big) + \sigma\big(\tilde{\eta}, [\lambda, \tilde{\xi}]\big).$$

This proves that $\mathrm{int}(\tilde{\xi})(d\Omega_{\tilde{\eta}}) - \mathrm{int}(\tilde{\eta})(d\Omega_{\tilde{\xi}}) = \Omega_{[\tilde{\xi}, \tilde{\eta}]} + d\big(\sigma(\tilde{\xi}, \tilde{\eta})\big)$, which then gives the desired identity in the first cohomology of the fibers of the Lagrangian fibration. $\square$

Remark 3.2. Since $\mathcal{V}^{1,0} \cong \mathscr{T}_{B^\circ}$ and $\mathcal{V}^{0,1} \cong \Omega^1_{B^\circ}$, this also implies that the Higgs field

$$\overline{\nabla} \colon \mathscr{T}_{B^\circ} \otimes \mathcal{V}^{1,0} \to \mathcal{V}^{0,1}$$

is symmetric in its two arguments. This fact is a special case of the "cubic condition" of Donagi and Markman [DM96]. The identity in Lemma 3.1 is actually equivalent to the statement that the hermitian metric on B° induced by the Hodge metric on the bundle $\mathcal{V}^{1,0}$ is a Kähler metric [Fre99].

The formula for the Lie bracket allows us to produce foliations from subvariations of V, using the fact that $F^1 \mathcal{V} \cong \mathscr{T}_{B^\circ}$ is the tangent bundle of B°.

Lemma 3.3. *Let $p \colon U \to B^\circ$ be a holomorphic mapping that is locally biholomorphic, and let W be a subvariation of Hodge structure of $V_U = p^* V$. Then the distribution given by the Hodge bundle $F^1 \mathcal{W} \subseteq F^1 \mathcal{V}_U$ is integrable, hence defines a foliation on U.*

Proof. Since p is locally biholomorphic, we have $F^1 \mathcal{V}_U = p^*(F^1 \mathcal{V}) \cong p^* \mathscr{T}_{B^\circ} \cong \mathscr{T}_U$. Because W is a subvariation of Hodge structure, the vector bundle $\mathcal{W}$ is preserved by the Gauss-Manin connection on $\mathcal{V}_U$. From $\mathcal{W} \cap F^1 \mathcal{V}_U = F^1 \mathcal{W}$, we get

$$[\xi, \eta] = \nabla_\xi(\eta) - \nabla_\eta(\xi) \in H^0(U, \mathcal{W}) \cap H^0(U, F^1 \mathcal{V}_U) = H^0(U, F^1 \mathcal{W})$$

for any two holomorphic vector fields ξ, η in the subspace $H^0(U, F^1 \mathcal{W}) \subseteq H^0(U, \mathscr{T}_U)$. This proves that the distribution is integrable. $\square$

We are also going to need the following simple fact about the integrability of the sum of two integrable distributions.

Lemma 3.4. *Let F and G be holomorphic foliations of rank d on a complex manifold, whose sum $F + G$ is a foliation of rank $2d$. Take a leaf L of F, and form the union of the leaves of G through points of L. Then the resulting submanifold is a leaf of $F + G$.*

Proof. Fix a point on L, and let S be the unique leaf of $F+G$ through that point; then S is a (locally defined) complex submanifold of dimension $2d$, and since $F \subseteq F + G$, it contains L. The restriction $G|_S$ is a foliation on S, and so S is locally the union of leaves of G. For dimension reasons, those leaves of $G|_S$ that pass through points of L fill up all of S, and this gives the desired result. $\square$

Remark 3.5. The proof of the lemma shows that the construction is actually symmetric in F and G: if we start from a leaf of G that is contained in S, and form the union of the leaves of F through its points, we get the same submanifold S as before.

Lastly, we need to analyze the behavior of the tangent bundle at smooth points of the discriminant locus D. It is known that D is always a divisor in B [HO09, Prop. 3.1], and that the local monodromy at any smooth point of D is always nontrivial [Hwa08, Prop. 3.2]. Our proof does not actually rely on these two facts, but it is nevertheless useful to keep them in mind. The next result says that if the local monodromy at a smooth point of D has finite order, then it looks like the local monodromy in an isotrivial elliptic fibration on a K3 surface.

Lemma 3.6. *Suppose that, in a neighborhood of a given point $b \in B$, the discriminant locus D is smooth of dimension $n - 1$, and that the local monodromy around D has finite order. Then the local monodromy transformation has exactly two nontrivial eigenvalues, which are a 4th or 6th root of unity and its complex conjugate.*

Proof. Let U be a neighborhood of the point in question, and suppose that $D \cap U$ is smooth and defined by a nontrivial holomorphic function $h \in \Gamma(U, \mathscr{O}_B)$. Let V_U denote the restriction of the variation of Hodge structure $V = R^1 f_*^\circ \mathbb{Q}_{X^\circ}$ to this open subset. By Matsushita's theorem, the isomorphism $\mathscr{T}_{U^\circ} \cong F^1 \mathcal{V}$ extends to an isomorphism

$$\mathscr{T}_U \cong F^1 \tilde{\mathcal{V}}_U^{>-1}$$

between the tangent bundle of U and the subbundle $F^1 \tilde{\mathcal{V}}_U^{>-1} = \tilde{\mathcal{V}}_U^{>-1} \cap j_*(F^1 \mathcal{V}_U)$ of the canonical extension of $(\mathcal{V}_U, \nabla)$ along the open embedding $j \colon U^\circ \to U$.

Since the monodromy transformation T has finite order, all of its eigenvalues are roots of unity. Moreover, the weight filtration defined by the logarithm of the unipotent part of T is trivial, and so the limiting mixed Hodge structure at b is a polarized Hodge structure of type $(1,0) + (0,1)$, and T is an automorphism of this Hodge structure [Sch73, Thm. 6.16]. By choosing a basis that is adapted to the Hodge decomposition and the eigenspace decomposition of T, we obtain a local frame

$$v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n} \in \Gamma\big(U, \tilde{\mathcal{V}}_U^{>-1}\big)$$

for the canonical extension, with the property that

$$\nabla(v_j) = \alpha_j \frac{dh}{h} \otimes v_j,$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{Q} \cap (-1, 0]$. As explained in [CK89, §2] (and in [SS22] for the case of complex variations of Hodge structure), we can then find a collection of holomorphic functions $f_{j,k}$ near the point b, such that the n sections

$$s_j = v_j + \sum_{k=n+1}^{2n} f_{j,k} v_k, \quad j = 1, \ldots, n,$$

define a local frame for the bundle $F^1 \tilde{\mathcal{V}}_U^{>-1}$, after shrinking the open set U if necessary. Because of the isomorphism $\mathscr{T}_U \cong F^1 \tilde{\mathcal{V}}_U^{>-1}$, we may also view $s_1, \ldots, s_n$ as holomorphic vector fields on U that generate the tangent bundle $\mathscr{T}_U$.

We now apply the formula for the Lie bracket in Lemma 3.1. It gives

$$[s_i, s_j] = \nabla_{s_i}(s_j) - \nabla_{s_j}(s_i) \equiv \alpha_j \frac{dh(s_i)}{h} \cdot s_j - \alpha_i \frac{dh(s_j)}{h} \cdot s_i \quad \mod v_{n+1}, \ldots, v_{2n},$$

and because of the special shape of the sections $s_1, \ldots, s_n$, the congruence is strong enough to imply that

$$[s_i, s_j] = \alpha_j \frac{dh(s_i)}{h} \cdot s_j - \alpha_i \frac{dh(s_j)}{h} \cdot s_i. \tag{9}$$

Now the left-hand side is again a holomorphic vector field on U, and so we deduce that, for every pair of indices $1 \le i < j \le n$, either $\alpha_i = 0$, or h divides $dh(s_j)$. This last condition means that the vector field s_j is tangent to the hypersurface $D \cap U$. Since not all the vector fields can be tangent to D, we conclude that at most one of $\alpha_1, \ldots, \alpha_n$ can be nonzero; in fact, exactly one of them has to be nonzero, because the monodromy transformation T is nontrivial.

To prove the remaining assertion, we observe that T is defined over $\mathbb{Z}$ and that $\det T = 1$. Therefore T has exactly two nontrivial eigenvalues, which are roots of unity and complex conjugates of each other. By checking the possible values of Euler's ϕ-function, it is then easy to see that the two eigenvalues must be either 4th or 6th roots of unity. $\square$

Remark 3.7. Suppose that $\alpha_1 \ne 0$ and $\alpha_2 = \cdots = \alpha_n = 0$. The identity in (9) shows that the $n-1$ holomorphic vector fields $s_2, \ldots, s_n$ are tangent to D and commute with each other. This observation will play an important role later on.

Remark 3.8. By the same method, one can show that if $T = T_s \cdot e^{2\pi i N}$ is the Jordan decomposition of the monodromy transformation (with $N^2 = 0$), then $\operatorname{rk} N \le 1$, and the limiting mixed Hodge structure contains a pure Hodge structure of weight 1 and dimension $\ge 2n - 2$. This is related to the classification of the general singular fiber of a Lagrangian fibration in [HO09].

4. Deformation theory and the universal family of curves

We first explain why Proposition 2.2 and Proposition 2.4, together with some basic deformation theory, imply that V splits off a factor universally computing the fixed part of the variation of Hodge structure on any rational curve of minimal degree, at least on a certain finite cover. We recommend Kollár's book [Kol96] for the necessary background on rational curves. Throughout, B is the smooth base of a Lagrangian fibration, B° is the complement of the discriminant locus, and $n = \dim B$.

Recall that $\omega_B^\vee = \det \mathscr{T}_B$ is ample [Mat99]. Suppose $g_0 : C_0 \to B$ is a rational curve meeting B° and of minimal anticanonical degree among such curves. We know from Proposition 2.2 that $g_0^* \mathscr{T}_B$ is semi-positive; consequently, we must have

$$\deg g_0^* \mathscr{T}_B = d + 1 \tag{10}$$

for some integer $d \in \{1, \ldots, n\}$. (If the degree is $\ge n + 2$, then Mori's Bend and Break technique produces a rational curve of smaller degree through every point of $g_0(C)$, contradicting minimality.) In view of Theorem 1.2, our goal will be to show that $d = n$. As B is Fano, it is known that a very general rational curve of minimal anticanonical degree deforms to cover B and moreover the pullback of the tangent bundle decomposes as

$$g_0^* \mathscr{T}_B \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^{\oplus(d-1)} \oplus \mathscr{O}^{\oplus(n-d)}. \tag{11}$$

By Proposition 2.4, the restriction of the variation of Hodge structure V to such a curve therefore has a fixed part of rank exactly $2(n - d)$.

We work with the (normalized) moduli space of rational curves $\mathrm{RatCurves}^n(B)$, following Kollár's notation [Kol96, II.2.5]. Let M be the open subset of $\mathrm{RatCurves}^n(B)$ consisting of all rational curves of anticanonical degree d that intersect B° and belong to the same connected component as g_0. The universal family $\pi \colon \mathcal{C} \to M$ is a $\mathbb{P}^1$-bundle over M, and we denote by $G \colon \mathcal{C} \to B$ the evaluation morphism. We use the notation $g \colon C \to B$ for the individual curves in our family; here C is the fiber of $\pi \colon \mathcal{C} \to M$ over the point in question, and $g = G|_C$.

For a point $g \colon C \to B$ in our moduli space M, we have an exact sequence

$$0 \to \mathscr{T}_C \xrightarrow{dg} g^* \mathscr{T}_B \to N_g \to 0$$

and the obstructions for M lie in $H^1(C, N_g)$. This group vanishes by Proposition 2.2, and so M is smooth at such a point, of dimension

$$\dim H^0(C, N_g) = \chi(C, N_g) = n + d - 2.$$

Moreover, as $g^* \mathscr{T}_B$ is globally generated and by the minimal degree condition, a generic deformation of our initial curve g_0 intersects the discriminant locus $D = B \setminus B^\circ$ transversely [Kol96, II.3.7]. Finally, for any $g \colon C \to B$ in the space M, and for any point $c \in C$, we have a short exact sequence

$$0 \longrightarrow H^0(C, \mathscr{T}_C) \xrightarrow{r_c - dg} T_c C \oplus H^0(C, g^* \mathscr{T}_B) \xrightarrow{\tau} T_c \mathcal{C} \longrightarrow 0$$

where r_c is restriction to c. The map τ is naturally interpreted as the derivative of the map $\nu \colon C \times \mathrm{Hom}(C, B) \to \mathcal{C}$ from the universal framed curve, and as the map $dg_c + r_c \colon T_c C \oplus H^0(C, g^* \mathscr{T}_B) \to T_{g(c)} B$ is the derivative of the universal framed map $G \circ \nu$ [Kol96, II.3.4], it follows that the induced map $T_c \mathcal{C} \to T_{g(c)} B$ is the derivative of G at the point $c \in \mathcal{C}$. Thus, again since $g^* \mathscr{T}_B$ is globally generated, the evaluation map $G \colon \mathcal{C} \to B$ is smooth at every point of $\mathcal{C}$.

Remark 4.1. The rational curves in our family are free, and so a generic curve $g \colon C \to B$ is an immersion [Kol96, II.3.14], but not necessarily an embedding. Of course, we will eventually show that $B \cong \mathbb{P}^n$ and that M is the space of lines in $\mathbb{P}^n$, but one has to be careful not to make any unjustified assumptions during the proof.

Let us summarize the conclusions that we have drawn from Proposition 2.2 and the deformation theory of rational curves.

Lemma 4.2. *The moduli space M is smooth of dimension $n + d - 2$, the universal curve $\pi \colon \mathcal{C} \to M$ is a $\mathbb{P}^1$-bundle over M, and the evaluation morphism $G \colon \mathcal{C} \to B$ is smooth of relative dimension $d - 1$.*

We denote by $G^\circ \colon \mathcal{C}^\circ \to B^\circ$ the base change to B°, and by $\pi^\circ \colon \mathcal{C}^\circ \to M$ the projection to the moduli space. By our choice of d, any rational curve of anticanonical degree d that meets B° must be irreducible and therefore cannot degenerate; this implies that G° is proper.

Now the general idea is the following. Suppose that $d < n$, which means that the variation of Hodge structure V on B° has a nontrivial fixed part on a general curve in our family. We are going to argue that there is a finite covering space of B°, constructed from the Stein factorization of $G^\circ \colon \mathcal{C}^\circ \to B^\circ$, on which the pullback of V decomposes in a nontrivial way. As a first step, we are going to construct such a decomposition on the universal curve $\mathcal{C}^\circ$ itself.

Consider the pullback $V_\mathcal{C} = (G^\circ)^* V$ of the variation of Hodge structure to $\mathcal{C}^\circ$, and its direct image $\pi_*^\circ V_\mathcal{C}$. By general theory, this is a constructible sheaf on M. Let $T \subseteq M$ be the (dense) Zariski open subset where this constructible sheaf is a local system (and therefore a variation of Hodge structure) of rank $2(n - d)$; this includes

all those curves $g\colon C \to B$ on which $g^* \mathscr{T}_B$ decomposes as in (11). (In fact, we will
show later that this is true for every curve, and so actually $T = M$; but this does not
matter for the time being.) By the semi-simplicity of polarized variations of Hodge
structure, we get a canonical splitting of rational variations of Hodge structure

$$V_{\mathcal{C}} \cong V_{\mathcal{C}}^{fix} \oplus W_{\mathcal{C}} \tag{12}$$

where $V_{\mathcal{C}}^{fix}$ is the unique subvariation of Hodge structure that agrees with the pullback
$(\pi^\circ)^*(\pi^\circ)_* V_{\mathcal{C}}$ on the open subset $\mathcal{C}_T = (\pi^\circ)^{-1}(T)$; this exists because the fundamental
group of this open subset surjects onto that of the complex manifold $\mathcal{C}^\circ$. Note that $V_{\mathcal{C}}^{fix}$
is itself a variation of Hodge structure of weight 1 and rank $2(n - d)$ that represents
the "universal" fixed part of the restriction to the fibers. In fact, the decomposition
in (12) descends to a finite covering of B°, as we now explain.

The fiber $G^{-1}(b)$ over a point $b \in B$, or rather its image in the moduli space M,
parametrizes all those rational curves in our family that pass through b. In the case of
projective space, the space of lines through a given point is irreducible, but this might
(a priori) not be the case here. We therefore consider the Stein factorization

$$
\begin{array}{ccc}
& \overset{G^\circ}{\overgroup{\hspace{4cm}}} & \\
\mathcal{C}^\circ \xrightarrow{\ s^\circ\ } & Z^\circ \xrightarrow{\ p^\circ\ } & B
\end{array}
$$

of the proper smooth morphism $G^\circ\colon \mathcal{C}^\circ \to B^\circ$; here s° is smooth with connected fibers
(of constant dimension $d - 1$), and p° is finite étale. Let $s\colon Z \to B$ be the unique
extension of this finite covering space to a finite morphism from a normal projective
variety Z. Because the universal curve $\mathcal{C}$ is smooth by Lemma 4.2, we get the following
commutative diagram:

$$
\begin{array}{ccc}
& \overset{G}{\overgroup{\hspace{4cm}}} & \\
\mathcal{C} \xrightarrow{\ s\ } & Z \xrightarrow{\ p\ } & B \\
\ \downarrow{\scriptstyle \pi} & & \\
M & &
\end{array} \tag{13}
$$

Remark 4.3. The fact that every rational curve $g\colon C \to B$ in our family factors through
the finite morphism $p\colon Z \to B$ means that the fundamental group of C° does not
generate the fundamental group of B° if the degree of p is positive, unlike in the case
of lines on projective space.

We again summarize all the relevant facts about the commutative diagram.

Lemma 4.4. *With the notation introduced above, the following is true:*

 (1) *The variety Z is smooth at every point of the open subset $s(\mathcal{C})$, and the map*
     $p\colon Z \to B$ *is étale there.*
 (2) *There is a splitting of variations of Hodge structure*

$$V_Z := (p^\circ)^* V = V_Z^{fix} \oplus W_Z \tag{14}$$

   of weight 1 that pulls back to (12).
 (3) *The decomposition in (14) induces a decomposition of the tangent bundle*

$$\mathscr{T}_{s(\mathcal{C})} \cong \mathscr{T}_{s(\mathcal{C})}^{fix} \oplus \mathscr{T}_{s(\mathcal{C})}^{pos}$$

   on $s(\mathcal{C})$, and the two summands define foliations of rank $n - d$ respectively d.
 (4) *All fibers of $\pi\colon \mathcal{C} \to M$ are tangent to the foliation $\mathscr{T}_{s(\mathcal{C})}^{pos}$.*

Proof. (1) follows from the fact that $G\colon \mathcal{C} \to B$ is smooth. For (2), observe that because $\mathcal{C}^\circ$ and Z° are smooth, the composition $\pi_1(\mathcal{C}_T^\circ) \to \pi_1(\mathcal{C}^\circ) \to \pi_1(Z^\circ)$ is surjective; this implies the claim. The restriction of $p\colon Z \to B$ to the open subset $s(\mathcal{C})$ is étale, and so

$$\mathcal{T}_{s(\mathcal{C})} \cong (p^* \mathcal{T}_B)|_{s(\mathcal{C})}.$$

Because $\mathcal{T}_B$ is the Hodge module extension of $F^1 \mathcal{V}$, it follows that $\mathcal{T}_{s(\mathcal{C})}$ is the Hodge module extension of $F^1 \mathcal{V}_Z$. The decomposition of the variation of Hodge structure therefore induces a decomposition of $\mathcal{T}_Z$ into a vector bundle of rank $n - d$ and a vector bundle of rank d. The two summands are foliations on $s(\mathcal{C})$ by Lemma 3.3. It remains to prove (4). By construction, the pullback of $\mathcal{T}_{s(\mathcal{C})}^{fix}$ to a general curve $g\colon C \to B$ in our moduli space is a trivial bundle of rank $n - d$. Since $g^* \mathcal{T}_B$ is always semipositive, it follows (by the deformation invariance of the degree) that the pullback of $\mathcal{T}_{s(\mathcal{C})}^{fix}$ is trivial for *every* curve in M. For degree reasons, $\mathcal{T}_C \cong \mathcal{O}(2)$ must therefore be contained in the pullback of the other summand, and so all curves in the universal family are indeed tangent to the foliation $\mathcal{T}_{s(\mathcal{C})}^{pos}$. $\square$

5. CONNECTED COMPONENTS OF THE FAMILY OF RATIONAL CURVES

We are going to need two additional results about the connected components of the family of rational curves through a given point. We keep the notation introduced in (13). At least generically, the fibers of $p\colon Z \to B$ represent the different connected components of the family of rational curves through a point. (This interpretation is slightly handicapped by the fact that a generic rational curve in our family can a priori have nodes.) Indeed, a point $z \in Z^\circ$ can be thought of as the point $p(z) \in B^\circ$, together with the connected component $s^{-1}(z)$ of the fiber $G^{-1}(p(z))$; of course, the connected component of the family of curves through b is really the image $M_z = \pi(s^{-1}(z))$ in the moduli space M. For any $z \in Z$, we denote by

$$P_z := s(\pi^{-1}(M_z)) \subseteq Z \tag{15}$$

the locus in Z swept out by the curves in the given component.[2] By construction, $z \in P_z$. The following result, which is mostly due to Araujo [Ara06], shows that P_z is generically a projective space.

Lemma 5.1. *For any $z \in Z^\circ$, the following is true:*

(1) *The subvariety $P_z \subseteq Z$ is isomorphic to $\mathbb{P}^d$, and its normal bundle in Z is a trivial bundle of rank $n - d$.*

(2) *At any point $z' \in P_z \cap Z^\circ$, the isomorphism*

$$T_{z'} Z \cong T_{p(z')} B \cong F^1 \mathcal{V}|_{p(z')} \cong F^1 \mathcal{V}_Z|_{z'}$$

maps the tangent space $T_{z'} P_z$ isomorphically to the fiber $F^1 \mathcal{W}_Z|_{z'}$.

(3) *Two subvarieties P_z and $P_{z'}$ are either equal or disjoint.*

Proof. Since $G^\circ \colon \mathcal{C}^\circ \to B^\circ$ is proper, it is clear that P_z is projective. Let us first show that $\dim P_z = d$. Since $\pi\colon \mathcal{C} \to M$ is a $\mathbb{P}^1$-bundle, the subvariety $s^{-1}(z)$ must be finite over its image M_z in M (because otherwise G would be constant along the curve in question). Therefore $\dim M_z = d - 1$ and $\dim \pi^{-1}(M_z) = d$. Now observe that the projection from $\pi^{-1}(M_z)$ to P_z is finite away from $p^{-1}(z)$. Indeed, if the fiber over a point $z' \in P_z$ with $p(z') \neq p(z)$ had positive dimension, we would get a nontrivial family of rational curves through the two fixed points $p(z)$ and $p(z')$, and by by Mori's Bend and Break technique, this would produce a rational curve through $p(z)$ of smaller anticanonical degree, contradicting our initial choice.

---

[2]This notation is different from Hwang's notation in [Hwa08].

By construction, we have $P_z \subseteq s(\mathcal{C})$. From Lemma 4.4, we get a decomposition

$$\mathscr{T}_{s(\mathcal{C})} \cong \mathscr{T}_{s(\mathcal{C})}^{fix} \oplus \mathscr{T}_{s(\mathcal{C})}^{pos},$$

and the summand $\mathscr{T}_{s(\mathcal{C})}^{pos}$ is a foliation of rank d. Since P_z is a union of curves that are tangent to this foliation, it follows that the tangent space to P_z at each point is contained in the fiber of $\mathscr{T}_{s(\mathcal{C})}^{pos}$. For dimension reasons, this means that P_z is smooth and $T_{z'}P_z = \mathscr{T}_{s(\mathcal{C})}^{pos}\big|_{z'}$ for every $z' \in P_z$, which gives (2). We also get

$$\mathscr{T}_{P_z} \cong \mathscr{T}_{s(\mathcal{C})}^{pos}\big|_{P_z} \quad \text{and} \quad N_{P_z|Z} \cong \mathscr{T}_{s(\mathcal{C})}^{fix}\big|_{P_z}.$$

Moreover, any two leaves P_z and $P_{z'}$ are either equal or disjoint, proving (3).

Now let $g\colon C \to B$ be any curve in the subset $M_z \subseteq M$, and denote by $i\colon C \to P_z$ the induced morphism to P_z. Because we already know that the restriction of $\mathscr{T}_{s(\mathcal{C})}^{fix}$ to every curve in M is trivial, we get

$$\deg i^* \mathscr{T}_{P_z} = \deg i^* \mathscr{T}_{s(\mathcal{C})} = \deg g^* \mathscr{T}_B = d + 1.$$

This means that there is a rational curve of anticanonical degree $d+1$ through any two points of P_z (namely the rational curves that sweep out the subvariety $P_{z'}$ for any $z' \in P_z$), and by our choice of d, these curves can obviously not degenerate. We can therefore apply [CMSB02, Thm. 0.1] and conclude that $P_z \cong \mathbb{P}^d$ and that the curves in the subset $M_z \subseteq M$ are exactly the lines through the point $z \in P_z$. Because $p\colon Z \to B$ is étale on the open subset $s(\mathcal{C})$, it also follows that

$$g^* \mathscr{T}_B \cong i^* \mathscr{T}_{P_z} \oplus \mathscr{O}^{\oplus(n-d)} \cong \mathscr{O}(2) \oplus \mathscr{O}^{\oplus(d-1)} \oplus \mathscr{O}^{\oplus(n-d)},$$

and so the pullback of the tangent bundle of B actually has the same splitting behavior as in (11) on *every* curve in the universal family.

Finally, the normal bundle $N_{P_z|Z}$ is a vector bundle of rank $n-d$ on $P_z \cong \mathbb{P}^d$ whose restriction to every line is trivial; by general facts about vector bundles on projective space, it follows that $N_{P_z|Z}$ is a trivial bundle of rank $n-d$, and (1) is proved. $\qquad\square$

Remark 5.2. One can use the results above to show that the open subset $s(\mathcal{C}) \subseteq Z$ is actually $\mathbb{P}^d$-bundle over a smooth variety of dimension $n-d$, whose fibers are the projective spaces P_z. The key point is that relevant Hilbert scheme of Z is smooth of dimension $n-d$ at the point corresponding to P_z, due to the fact that the normal bundle of P_z in Z is trivial. For details, see [Ara06, Lem. 3.3].

The second result is a proposition proved by Hwang in his earlier paper [Hwa07]. As it plays an important role in our argument, we include the proof. First a lemma.

Lemma 5.3. *Let $z_1, z_2 \in Z^\circ$ be two points. If $p(P_{z_1}) = p(P_{z_2})$, then $P_{z_1} = P_{z_2}$.*

Proof. Suppose that $p(P_{z_1}) = p(P_{z_2})$, and denote by P the normalization of this sub-variety of B. Since $P_{z_j} \subseteq s(\mathcal{C})$, the induced morphism $P_{z_j} \to P$ is a finite immersion, and therefore a finite covering space. But $P_{z_j} \cong \mathbb{P}^d$, and therefore $P \cong \mathbb{P}^d$; this is simply connected, and so $P_{z_j} \to P$ is an isomorphism. By composing one isomorphism with the inverse of the other, we get an isomorphism between P_{z_1} and P_{z_2}. Under the isomorphism $P_{z_j} \cong \mathbb{P}^d$, any line in $\mathbb{P}^d$ determines a rational curve in B of anticanonical degree $d+1$. Since every automorphism of $\mathbb{P}^d$ takes lines to lines, it follows that $\pi\big(s^{-1}(P_{z_1})\big) = \pi\big(s^{-1}(P_{z_2})\big)$, and therefore $P_{z_1} = P_{z_2}$. $\qquad\square$

The following result is a slight generalization of [Hwa07, Prop. 2.2].

Proposition 5.4. *Let $z_1, z_2 \in Z^\circ$ be two points with $p(z_1) = p(z_2) = b$. If the subspaces*

$$F^1 \mathcal{W}_Z|_{z_1} \subseteq F^1 \mathcal{V}_Z|_{z_1} \cong F^1 \mathcal{V}|_b \quad \text{and} \quad F^1 \mathcal{W}_Z|_{z_2} \subseteq F^1 \mathcal{V}_Z|_{z_2} \cong F^1 \mathcal{V}|_b$$

have a nontrivial intersection in $F^1\mathcal{V}|_b$, then $P_{z_1} = P_{z_2}$.

Proof. Since $p\colon Z \to B$ is étale at the point z_1, we can find an open neighborhood $U \subseteq B$ of the point b, and and open neighborhood $U_1 \subseteq Z$ of the point z_1, such that $p|_{U_1}\colon U_1 \to U$ is biholomorphic. The pullback of $W_Z|_{U_1}$ along the inverse of this biholomorphic mapping therefore defines a subvariation of Hodge structure W_1 of $V|_U$, whose rank is $2d$. After applying the same construction to the point z_2, we obtain another subvariation of Hodge structure W_2 of $V|_U$, also of rank $2d$. Since the two Hodge bundles $F^1\mathcal{W}_1$ and $F^1\mathcal{W}_2$ have a nontrivial intersection at the point b, the intersection $W = W_1 \cap W_2$ must be a nontrivial subvariation. According to Lemma 3.3, the intersection $F^1\mathcal{W}_1 \cap F^1\mathcal{W}_2 = F^1\mathcal{W}$ is therefore a foliation of positive rank on U. The leaf of the foliation through the point b is contained in both $p(P_{z_1})$ and $p(P_{z_2})$, and so the intersection $p(P_{z_1}) \cap p(P_{z_2})$ must have positive dimension.

As $P_z \to p(P_z)$ is always birational, we can therefore find a smooth projective curve Y and two morphisms $f_1\colon Y \to P_{z_1}$ and $f_2\colon Y \to P_{z_2}$ such that $p \circ f_1 = p \circ f_2$. We can clearly arrange that the image of Y passes through the point b. The following commutative diagram shows the relevant morphisms:

$$
\begin{array}{ccc}
Y & \xrightarrow{\ f_1\ } & P_{z_1} \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle p} \\
P_{z_2} & \xrightarrow{\ p\ } & B
\end{array}
$$

From Lemma 5.1, we know that $p^*\mathscr{T}_B|_{P_z} \cong \mathscr{T}_{P_z} \oplus \mathscr{O}^{\oplus(n-d)}$, and that the tangent bundle to $P_z \cong \mathbb{P}^d$ is the restriction of the bundle $\mathscr{T}^{pos}_{s(\mathcal{C})}$. Since the tangent bundle of projective space is ample, the identity

$$
f_1^*\mathscr{T}_{P_{z_1}} \oplus \mathscr{O}^{\oplus(n-d)} = f_2^*\mathscr{T}_{P_{z_2}} \oplus \mathscr{O}^{\oplus(n-d)}
$$

inside the pullback of $\mathscr{T}_B$ clearly implies that $f_1^*\mathscr{T}_{P_{z_1}}$ and $f_2^*\mathscr{T}_{P_{z_2}}$ are the same subbundle. But this means that $F^1\mathcal{W}_Z|_{z_1}$ and $F^1\mathcal{W}_Z|_{z_2}$ are equal, as subspaces of $F^1\mathcal{V}|_b$, from which it follows as above that $p(P_{z_1}) \cap p(P_{z_2})$ has dimension d, and hence that $p(P_{z_1}) = p(P_{z_2})$. We now reach the desired conclusion by applying Lemma 5.3. $\qquad\square$

6. Controlling the ramification of the finite covering

After the results in the four preceding sections, the proof of Hwang's theorem is reduced to showing that the splitting in Lemma 4.4 descends to B° (where it contradicts the irreducibility of V). Our proof of this fact is a version of [Hwa08, §5], but becomes simpler because everything is controlled by variations of Hodge structure. The crucial step is to control the ramification of the finite morphism $p\colon Z \to B$ via the following result.

Proposition 6.1. *Let $g\colon C \to B$ be the rational curve corresponding to a general point in the moduli space M. Then the base change of the finite covering space $p^\circ\colon Z^\circ \to B^\circ$ along $g^\circ\colon C^\circ \to B^\circ$ is a disjoint union of copies of C°.*

Let $g\colon C \to B$ be a generic rational curve (in the universal family $\mathcal{C}$). Such a curve intersects the discriminant locus D transversely (because all the rational curves under consideration are free), and as Z is normal, the base change

$$
\begin{array}{ccc}
C \times_B Z & \longrightarrow & Z \\
\downarrow & & \downarrow{\scriptstyle p} \\
C & \xrightarrow{\ g\ } & B
\end{array}
$$

is a finite union of smooth projective curves, each with a finite morphism to C. Since $G\colon \mathcal{C} \to B$ factors through $p\colon Z \to B$ by construction, exactly one of these curves is (the image of) $C \subseteq \mathcal{C}$. Let $p\colon R \to C$ be one of the other components. If $\deg(R/C) = 1$, then p is an isomorphism, and so R° is isomorphic to C°. In order to prove Proposition 6.1, we may therefore assume from now on that $\deg(R/C) \geq 2$; after several steps, we will see that this assumption leads to a contradiction. The idea is the same as in Hwang's paper: to first reduce to the $d = 1$ case (so the curves in M don't deform with a fixed point), and then to show there is a deformation of C for which $R \to C$ deforms trivially but for which constant sections deform along curves in the family M. The condition $\deg(R/C) \geq 2$ then contradicts Proposition 5.4.

Lemma 6.2. If $\deg(R/C) \geq 2$, then the pullback of the variation of Hodge structure V to R° is constant.

Proof. We use the superscript "$\circ$" to denote the base change to the open subset B°. We have the following commutative diagram:

$$\begin{array}{ccc} R^\circ & \longrightarrow & Z^\circ \\ \downarrow{\scriptstyle p^\circ} & & \downarrow{\scriptstyle p^\circ} \\ C^\circ & \xrightarrow{\ g^\circ\ } & B^\circ \end{array}$$

We obtain two decompositions as variations of Hodge structure of $V_R := V_Z|_{R^\circ}$. One decomposition $V_R \cong V_R^{fix} \oplus W_R$ comes from restricting (14) to R°; the other comes from pulling back $V_C = (g^\circ)^*V \cong V_C^{fix} \oplus W_C$ along the map $p^\circ\colon R^\circ \to C^\circ$. Note that both W_C and W_R are variations of Hodge structure of rank $2d$, where d is the integer from (10). We now play the two decompositions off against each other.

Consider the intersection $(g^\circ)^*W_C \cap W_R$, which is again a subvariation of Hodge structure of V_R. If it is nontrivial, then the intersection of the two Hodge bundles $(g^\circ)^*F^1\mathcal{W}_C$ and $F^1\mathcal{W}_R$ is a vector bundle of positive rank on all of R°. According to Proposition 5.4, this gives us $P_{\tilde{g}(y)} = P$ for every $y \in R^\circ$, where P is the projective space containing our original curve. But this is clearly impossible. The conclusion is that $(g^\circ)^*W_C$ and W_R intersect trivially. This makes the composition

$$W_R \to V_R \to (p^\circ)^*V_C^{fix}$$

of the inclusion and the projection injective. As V_C^{fix} is a constant variation of Hodge structure on C°, it follows that W_R is constant, too.

Now we claim that the intersection of the subvariations W_R and $(p^\circ)^*V_C^{fix}$ is also trivial. Otherwise, the Hodge bundle $F^1\mathcal{W}_R$ contains a subbundle of positive rank that is the pullback of a trivial bundle from C°, which means that the two subspaces $F^1\mathcal{W}_R|_y$ and $F^1\mathcal{W}_R|_{y'}$ intersect inside $F^1\mathcal{V}_{p(y)}$, for any two points $y, y' \in R^\circ$ such that $p(y) = p(y')$. By Proposition 5.4, it follows that $P_{\tilde{g}(y)} = P_{\tilde{g}(y')}$, and as $\deg(R/C) \geq 2$, this is clearly impossible. The conclusion is that, for dimension reasons,

$$V_R \cong W_R \oplus (p^\circ)^*V_C^{fix}$$

is the sum of two constant variations of Hodge structure on R. $\qquad\square$

Remark 6.3. Lemma 6.2 in particular implies there is a covering family of curves on which V is isotrivial, which contradicts the main result of [Bak22] (and therefore proves Proposition 6.1) unless the Lagrangian fibration is isotrivial. In the isotrivial case the foliations all come from a flat structure on $\mathcal{T}_{B^\circ}$, and are therefore easier to think about, but we do not in fact use this reduction.

The next step is to show that $d = 1$.

Lemma 6.4. *In the above situation, we must have $d = 1$.*

Proof. Because $p\colon R \to C$ is étale over C°, it must ramify over at least one point $x \in C$ such that $g(x) \in D$. Being generic, $g\colon C \to B$ intersects the discriminant locus D transversely, and so $g(x)$ is a smooth point of D. The pullback of V to R° has trivial monodromy (by Lemma 6.2), hence the Hodge structures do not vary from point to point; as C° is quasi-projective, this implies that the Hodge bundle $F^1\mathcal{V}_C$ is a flat subbundle of $\mathcal{V}_C$. It also follows that the local monodromy of V_C around the point x has finite order; consequently, the local monodromy of V near $g(x)$ has finite order as well. According to Lemma 3.6, the monodromy transformation therefore has exactly two nontrivial eigenvalues, and because the Hodge bundle $F^1\mathcal{V}_C$ is a flat subbundle of $\mathcal{V}_C$, it contains (in a punctured neighborhood of the point x) a subbundle of rank $n - 1$ that is preserved by the monodromy transformation around x. Now if $d \geq 2$, then it follows as above that at any two points $y, y' \in R^\circ$ with $p(y) = p(y')$ sufficiently close to x, the two subspaces $F^1\mathcal{W}_R|_y$ and $F^1\mathcal{W}_R|_{y'}$ intersect nontrivially. As this is impossible (by Proposition 5.4), we conclude that $d = 1$. $\qquad\square$

This reduces the problem to the case $d = 1$. The smooth morphism $G\colon \mathcal{C} \to B$ now has finite fibers, and so $s\colon \mathcal{C} \to Z$ is an open embedding. We may therefore identify the universal curve $\mathcal{C}$ with an open subset of Z for the remainder of the argument. We have $\dim \mathcal{C} = n$ and $\dim M = n - 1$; because there is nothing to prove when $n = 1$, we shall assume from now on that $n \geq 2$.

Recall from Lemma 4.2 that $p\colon Z \to B$ is étale at every point of $\mathcal{C}$. According to Lemma 4.4, we have a decomposition of the tangent bundle

$$\mathscr{T}_\mathcal{C} \cong \mathscr{T}_\mathcal{C}^{fix} \oplus \mathscr{T}_\mathcal{C}^{pos} \tag{16}$$

into two foliations of rank $n - 1$ respectively 1, induced by the decomposition

$$V_Z = V_Z^{fix} \oplus W_Z$$

of the variation of Hodge structure $V_Z = (p^\circ)^* V$ on Z°. From the construction in (12), it is clear that $\mathscr{T}_\mathcal{C}^{fix}$ is the pullback of a vector bundle of rank $n - 1$ from the moduli space M, whereas $\mathscr{T}_\mathcal{C}^{pos}$ is tangent to the fibers of $\pi\colon \mathcal{C} \to M$. By comparing (16) with the short exact sequence for the relative tangent bundle

$$0 \to \mathscr{T}_{\mathcal{C}/M} \to \mathscr{T}_\mathcal{C} \to \pi^* \mathscr{T}_M \to 0,$$

we deduce that $\mathscr{T}_\mathcal{C}^{fix} \cong \pi^* \mathscr{T}_M$ and $\mathscr{T}_\mathcal{C}^{pos} \cong \mathscr{T}_{\mathcal{C}/M}$.

Denote by $\tilde{g}\colon R \to Z$ the morphism induced by $p\colon R \to B$, as in the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\tilde{g}} & Z \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ C & \xrightarrow{g} & B \end{array}$$

The intersection $R \cap \tilde{g}^{-1}(\mathcal{C})$ is a dense Zariski open subset of the curve R that contains R°. Under the projection $\pi\colon \mathcal{C} \to M$, which is proper, the image $\pi(\tilde{g}(R) \cap \mathcal{C})$ is a quasi-projective curve inside the moduli space M.

Lemma 6.5. *The morphism $\pi \circ \tilde{g}\colon R \cap \tilde{g}^{-1}(\mathcal{C}) \to \pi(\tilde{g}(R) \cap \mathcal{C})$ is an immersion.*

Proof. Suppose to the contrary that $y \in R \cap \tilde{g}^{-1}(\mathcal{C})$ is a branch point. Set $z = \tilde{g}(y) \in \tilde{g}(R) \cap \mathcal{C}$. This means that $\tilde{g}(R)$ is tangent to the fiber $P_z = \pi^{-1}\big(\pi(z)\big)$ at the point z. Since $p\colon Z \to B$ is étale there, we have $T_z Z \cong T_{p(z)} B$. Under this isomorphism, the tangent space $T_y R$ maps isomorphically to $T_{p(z)} C$, and the tangent space $T_z P_z$ is isomorphic to $\mathscr{T}_\mathcal{C}^{pos}|_z$. By Lemma 6.2, the variation of Hodge structure $(p^\circ)^* V_C$ on R°

is constant. It therefore extends uniquely to a constant variation of Hodge structure V_R on the entire curve R, and both $W_Z|_{R^\circ}$ and $(p^\circ)^* W_C$ extend to constant subvariations of rank 2. The condition that $T_z Z$ and $T_y R$ have the same image inside $T_{p(z)} B$ means that these two subvariations are equal at the point y. Being constant, they must then be equal everywhere, and as usual, this is in contradiction with Proposition 5.4. $\qquad\square$

We now investigate what happens when we deform the rational curve $g \colon C \to B$. Because $\pi \colon \mathcal{C} \to M$ is a $\mathbb{P}^1$-bundle, we can choose a small open neighborhood M_0 of the point $g \colon C \to B$ in the moduli space M, say with M_0 biholomorphic to an open ball, such that $\pi^{-1}(M_0) \cong M_0 \times \mathbb{P}^1$. The original rational curve is now a morphism $g \colon \mathbb{P}^1 \to B$, and the nearby subvarieties P_z are the different copies of $\mathbb{P}^1$. The following diagram shows the relevant morphisms:

$$
\begin{array}{ccc}
M_0 \times \mathbb{P}^1 & \xrightarrow{\;G\;} & B \\
\downarrow{\scriptstyle\pi} & & \\
M_0 & &
\end{array}
$$

By restricting the decomposition in Lemma 4.4 to the open subset $M_0 \times \mathbb{P}^1$, we obtain a decomposition of the tangent bundle

$$
\mathscr{T}_{M_0 \times \mathbb{P}^1} \cong \mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1} \oplus \mathscr{T}^{pos}_{M_0 \times \mathbb{P}^1}
$$

in which the two summands are foliations of rank $n-1$ respectively 1. By construction, the fibers of π are leaves of $\mathscr{T}^{pos}_{M_0 \times \mathbb{P}^1}$, and the restriction of $\mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1}$ to the fibers is trivial.

Since the initial curve $g \colon \mathbb{P}^1 \to B$ intersects D transversely, the inverse image $G^{-1}(D)$ is a union of finitely many sections $(\mathrm{id} \times h_j)(M_0)$, where $h_1, \ldots, h_m \colon M_0 \to \mathbb{P}^1$ are holomorphic mappings. The next result says that each of these sections is actually a leaf of the foliation $\mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1}$.

Lemma 6.6. *The foliation $\mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1}$ is tangent to the divisor $G^{-1}(D)$.*

Proof. The argument is very similar to the proof of Lemma 3.6. The holomorphic mapping from $M_0 \times \mathbb{P}^1 \to Z$ is an open embedding, and by Lemma 4.4, the tangent bundle $\mathscr{T}_{M_0 \times \mathbb{P}^1}$ is therefore the canonical extension of the pullback of the Hodge bundle $F^1 \mathcal{V}_Z$. Since $V_Z = V_Z^{fix} \oplus W_Z$, we can apply the construction in Lemma 3.6 in a neighborhood of any component of $G^{-1}(D)$, but choosing the sections v_1 and v_{n+1} in the canonical extension of W_Z, and the remaining sections $v_2, \ldots, v_n$ and $v_{n+2}, \ldots, v_{2n}$ in the canonical extension of V_Z^{fix}. Then s_1 is a local frame for the canonical extension of $F^1 \mathcal{W}_Z$, and $s_2, \ldots, s_n$ are a local frame for the canonical extension of $F^1 \mathcal{V}_Z^{fix}$, and therefore for the foliation $\mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1}$. The same argument as before shows that $s_2, \ldots, s_n$ are tangent to the given component of $G^{-1}(D)$, and this proves the claim. $\qquad\square$

The tangent bundle of $M_0 \times \mathbb{P}^1$ is globally generated (because $\mathscr{T}_{\mathbb{P}^1}$ is globally generated and M_0 is a Stein manifold); the same thing is therefore true for the foliation $\mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1}$. After composing G with an automorphism of $M_0 \times \mathbb{P}^1$, we can arrange that the three standard sections $M_0 \times \{0\}$, $M_0 \times \{1\}$, and $M_0 \times \{\infty\}$ are leaves of the foliation $\mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1}$. From the isomorphism

$$
H^0 \big(M_0 \times \mathbb{P}^1, \mathscr{T}_{M_0 \times \mathbb{P}^1} \big) \cong H^0(M_0, \mathscr{T}_{M_0}) \oplus H^0(M_0, \mathscr{O}_{M_0}) \otimes H^0 \big(\mathbb{P}^1, \mathscr{T}_{\mathbb{P}^1} \big)
$$

we then conclude that *every* leaf of $\mathscr{T}^{fix}_{M_0 \times \mathbb{P}^1}$ must be of the form $M_0 \times \{x\}$ for a point $x \in \mathbb{P}^1$. In particular, every component of the divisor $G^{-1}(D)$ must be of this form, which means that the holomorphic mappings $h_1, \ldots, h_m \colon M_0 \to \mathbb{P}^1$ are constant.

From this fact, we can easily deduce that the finite covering $p \colon R \to \mathbb{P}^1$ does not change as we deform $g \colon \mathbb{P}^1 \to B$. Recall that R is a smooth projective curve, and that $\tilde{g} \colon R \to Z$ is the induced morphism. After shrinking M_0, if necessary, the base change

$$
\begin{array}{ccc}
(M_0 \times \mathbb{P}^1) \times_B Z & \longrightarrow & Z \\
\downarrow & & \downarrow{\scriptstyle p} \\
M_0 \times \mathbb{P}^1 & \xrightarrow{\ G\ } & B
\end{array}
$$

is also smooth; let $\mathcal{R}$ be the unique component containing R. We have just shown that $\mathcal{R}^\circ \to M_0 \times (\mathbb{P}^1)^\circ$ is a finite covering space; because finite covering spaces of a fixed quasi-projective curve do not deform in a nontrivial way, we get $\mathcal{R} \cong M_0 \times R$. The following commutative diagram shows the relevant morphisms:

$$
\begin{array}{ccc}
 & & \mathcal{C} \xrightarrow{\ \pi\ } M \\
 & & \uparrow \\
M_0 \times R & \xrightarrow{\ \tilde{G}\ } & Z \\
\downarrow{\scriptstyle \mathrm{id}\,\times p} & & \downarrow{\scriptstyle p} \\
M_0 \times \mathbb{P}^1 & \xrightarrow{\ G\ } & B
\end{array}
$$

According to Lemma 6.5, the morphism

$$
\pi \circ \tilde{g} \colon R \cap \tilde{g}^{-1}(\mathcal{C}) \to \pi\big(\tilde{g}(R) \cap \mathcal{C}\big)
$$

is unramified. Because unramified morphisms between two fixed quasi-projective curves also do not deform in a nontrivial way, the following lemma will very quickly lead to the desired contradiction.

Lemma 6.7. *There is a 1-dimensional family of deformations of the initial rational curve $g \colon \mathbb{P}^1 \to B$ for which the curves $\pi(\tilde{g}(R) \cap \mathcal{C}) \subseteq M$ stay the same.*

Proof. Consider the quasi-projective curve $\pi(\tilde{g}(R) \cap \mathcal{C})$ inside the moduli space M. Its preimage under π is a quasi-projective surface in $\mathcal{C}$. Since $p \colon \mathcal{C} \to B$ is smooth of relative dimension 0, it follows that

$$
S = p\Big(\pi^{-1}\big(\pi(\tilde{g}(R) \cap \mathcal{C})\big)\Big) \subseteq B \tag{17}
$$

is a quasi-projective surface in B that contains the intersection $g(\mathbb{P}^1) \cap B^\circ$. A priori, S could be singular along this curve, but we will show in a moment that it is actually smooth in a neighborhood of $g(\mathbb{P}^1) \cap B^\circ$. We can think of S as being swept out by those rational curves in our family that intersect R.

Let us first prove that S is smooth at every point $b_0 \in g(\mathbb{P}^1) \cap B^\circ$. Let $z_0 \in \mathcal{C}^\circ$ be a point on the curve $g \colon \mathbb{P}^1 \to B$ such that $p(z_0) = b_0$, and enumerate the points in the fiber $p^{-1}(b_0) \cap R^\circ$ as $y_1, \ldots, y_r$, where $r = \deg(R/\mathbb{P}^1)$; set $z_j = \tilde{g}(y_j)$. Choose an open neighborhood $U \subseteq B$ of the point b_0, and an open neighborhood $U_j \subseteq \mathcal{C}$ of each point z_j, such that $p|_{U_j} \colon U_j \to U$ is a biholomorphism. Pulling back the $r + 1$ variations of Hodge structure $W_Z|_{U_j}$, we get $r + 1$ subvariations $W_0, \ldots, W_r \subseteq V|_U$. From the proof of Lemma 6.2, we know that the subvariation $W_R + (p^\circ)^* W_C$ of V_R has rank $4d = 4$, and that its fibers at any two points $y, y' \in R^\circ$ with $p(y) = p(y')$ map to the same subspace of $T_{p(y)} B$. This implies that $W = W_0 + W_1 + \cdots + W_r$ has rank 4 at the point b_0, and therefore on all of U. Consequently, the Hodge bundle $F^1 W$ defines a foliation of rank 2 on the open set U, and this foliation is the sum of the two foliations $F^1 W_0$ and $F^1 W_j$ of rank 1, for every $j = 1, \ldots, r$.

From (17), it is easy to see that all those curves involved in the construction of S that intersect the open subset U must be tangent to the foliation $F^1\mathcal{W}$. For dimension reasons, this implies that $S \cap U$ is the leaf of the foliation $F^1\mathcal{W}$ through the point b_0, and therefore smooth.

Now let C_0 denote the branch of the curve $g \colon \mathbb{P}^1 \to B$ corresponding to the point z_0; to be precise, we have $C_0 = p(U_0 \cap P_{z_0})$. Similarly, let $C_j = p(U_j \cap P_{z_j})$ be the branches of the other r rational curves. By construction, each C_j is the leaf of the foliation $F^1\mathcal{W}_j$ through b_0. According to Lemma 3.4, the surface germ $S \cap U$ is therefore swept out by the deformations of the curve germ C_0 along each of the curves C_j. This gives us the desired 1-dimensional family of deformations of the rational curve $g \colon \mathbb{P}^1 \to B$. $\square$

We can now finish the proof of Proposition 6.1 as follows. Let Δ be the unit disk, and represent the 1-dimensional family of deformations of $g \colon \mathbb{P}^1 \to B$ by a holomorphic mapping $h \colon \Delta \to M_0$ with $h(0) = [g]$. Let us denote by $g_t = G(h(t), -) \colon \mathbb{P}^1 \to B$ the rational curves in this family, and by $\tilde{g}_t = \tilde{G}(h(t), -) \colon R \to Z$ the resulting family of morphisms from R. By Lemma 6.5, the composition

$$\pi \circ \tilde{g}_t \colon R \cap \tilde{g}_t^{-1}(\mathcal{C}) \to \pi\big(\tilde{g}_t(R) \cap \mathcal{C}\big)$$

is unramified for every $t \in \Delta$. Since the source and the target stay fixed, it follows that $\pi \circ \tilde{g}_t$ is independent of t. This means that if we let $y_1, \ldots, y_r$ be the points in the fiber $p^{-1}(x)$ over a fixed point $x \in (\mathbb{P}^1)^\circ$, and set $z_j = \tilde{g}(y_j)$, then as $t \in \Delta$ varies, the image $\tilde{g}_t(y_j)$ sweeps out a small open subset of the curve P_{z_j}. Because $p \circ \tilde{g} = g \circ p$, it follows that $p(P_{z_1}) = \cdots = p(P_{z_r})$; but then Lemma 5.3 gives $P_{z_1} = \cdots = P_{z_r}$, and this clearly contradicts our initial assumption that $r \geq 2$.

7. Proof of the main theorem

At this point, we can prove the main theorem very quickly by putting together the results from the previous five sections.

Proof of Theorem 1.1. By way of contradiction, assume B is not isomorphic to $\mathbb{P}^n$. By Theorem 1.2, there must be a minimal degree rational curve $g_0 \colon C_0 \to B$ meeting B° with $d + 1 = -\deg g_0^* \mathscr{T}_B \leq n$. By Proposition 2.2, since $g_0^* \mathscr{T}_B$ contains a copy of $\mathscr{T}_{C_0} \cong \mathscr{O}(2)$, it follows that $g_0^* \mathscr{T}_B$ has a trivial factor, so by Proposition 2.4 and Lemma 4.4, we obtain a nontrivial splitting of the variation of Hodge structure V_Z. A general curve $g \colon C \to B$ meets the boundary transversely, and since $b_2(B) = 1$, it must meet every irreducible component of the boundary that has dimension $n - 1$. By Proposition 6.1 it follows that $p \colon Z \to B$ is unramified in codimension 1, and because Z is normal by construction, p must be an étale cover by purity of the branch locus [Nag59]. Since B is simply connected, it follows that p is an isomorphism; this means that there is a nontrivial splitting of V itself, which contradicts Voisin's result in Theorem 1.3. $\square$

Remark 7.1. The fact that M was a full deformation space of a family of rational curves admitting a splitting was crucial to the argument. Indeed, consider $S \to \mathbb{P}^1$ an elliptic K3 surface, and $X = S^{[n]} \to B = \mathbb{P}^n$ the Hilbert scheme of n points. Then taking the cover $\mathbb{P}^1 \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ obtained by quotienting $(\mathbb{P}^1)^n$ by S_{n-1}, we have a covering family of rational curves (namely the $\mathbb{P}^1$ fibers) along which the pullback of V has a fixed part (of rank $2n - 2$), and the cover $\mathbb{P}^1 \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ splits completely over the open part of these curves. These curves are all tangent to the diagonal of $\mathbb{P}^n$ however, so this does not imply $\mathbb{P}^1 \times \mathbb{P}^{n-1} \to \mathbb{P}^n$ is unramified (and indeed it is not), and it also does not imply that the splitting descends (and indeed it does not).

Remark 7.2. The proof of Theorem 1.1 works equally well for a Lagrangian fibration $f : X \to B$ of a primitive symplectic variety X, provided we additionally assume X is simply connected and that the discriminant is divisorial.

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