The Kodaira dimension of complex hyperbolic manifolds with cusps

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Abstract

We prove a bound relating the volume of a curve near a cusp in a complex ball quotient $X = \mathbb{B}/\Gamma$ to its multiplicity at the cusp. There are a number of consequences: we show that for an n-dimensional toroidal compactification $\overline{X}$ with boundary D, $K_{\overline{X}}+(1-\lambda)D$ is ample for $\lambda \in \left(0, \frac{n+1}{2\pi}\right)$, and in particular that $K_{\overline{X}}$ is ample for $n \geqslant 6$. By an independent algebraic argument, we prove that every ball quotient of dimension $n \geqslant 4$ is of general type, and conclude that the phenomenon famously exhibited by Hirzebruch in dimension 2 does not occur in higher dimensions. Finally, we investigate the applications to the problem of bounding the number of cusps and to the Green–Griffiths conjecture.

1. Introduction

Complex hyperbolic manifolds are complex manifolds admitting a complete finite-volume metric of constant negative sectional curvature. Such manifolds are quotients of the complex hyperbolic ball $\mathbb{B}$ by a discrete group of holomorphic isometries. On the one hand, just as for real hyperbolic manifolds, the topology of the uniformizing group is a powerful tool in studying their geometry. On the other hand, work of [AMRT75] and [Mok12] shows that such manifolds always admit orbifold toroidal compactifications whose algebraic geometry provides an equally powerful complementary set of techniques.

Quotients by arithmetic lattices naturally arise as Shimura varieties parametrizing abelian varieties with certain endomorphism structure, but many other interesting moduli spaces admit complex ball uniformizations: moduli spaces of low genus curves, del Pezzo surfaces, certain K3 surfaces (see *e.g.* [DK07] for an overview), and cubic threefolds [ACT11], to name a few. Importantly, the complex ball is the only bounded symmetric domain that admits nonarithmetic lattices [Mar84], and examples have only been constructed in dimensions 2 and 3 by Mostow [Mos80] and Deligne–Mostow [DM93] as period domains of hypergeometric differential forms. Many of the techniques available to study the geometry of ball quotients only apply in the arithmetic case, and consequently much less is known about nonarithmetic quotients.

In this paper we study curves in non-compact complex hyperbolic manifolds. Our first main result is:

Theorem A. *Let X be a complex hyperbolic manifold of dimension n whose toroidal compactification $\overline{X}$ has no orbifold points. Then $K_{\overline{X}} + (1-\lambda)D$ is ample for $0 < \lambda < \frac{n+1}{2\pi}$.*

Corollary B. *With the above assumptions, $K_{\overline{X}}$ is ample provided $n \geqslant 6$. Thus, $\overline{X}$ is the canonical model of X.*

2010 Mathematics Subject Classification 32Q45 (Primary); 14C20, 22E40 (Secondary).
Keywords: hyperbolic manifold, ball quotient, Kodaira dimension, toroidal compactification.

Of course, if X is already compact, then K_X is clearly ample. Theorem A is a special case of the more refined positivity statements in Propositions 3.3 and 3.6. To give an idea for the general result, we show $(K_{\overline{X}}+D)-\frac{n+1}{4\pi}\sum_i s_iD_i$ is ample if the depth s_i horoball neighborhoods V_i of the cusps are embedded and disjoint. This is a condition that can be understood in terms of the uniformizing group Γ. Theorem A is proven by showing that the volume of a curve in V_i is bounded by its multiplicity along the corresponding boundary divisor with coefficient depending on the depth s_i. Parker's generalization of Shimizu's lemma [Par98] bounds the minimal depth of the cusps of a discrete group Γ, but better bounds on the ample cone can be extracted from specific knowledge of the parabolic subgroups. The more general results apply to the orbifold case as well.

The toroidal compactification of a complex hyperbolic manifold satisfies the hypotheses of Theorem A under mild assumptions on the uniformizing group (see Definition 2.3), and every complex hyperbolic orbifold admits a finite étale cover which satisfies this property. Note that $K_{\overline{X}}+D$ induces the contraction $\overline{X}\to X^*$ to the Baily–Borel compactification, and therefore always generates one of the boundary rays of the slice of the nef cone cut out by the plane generated by $K_{\overline{X}}$ and D. It is an interesting question in general for toroidal compactifications (not necessarily of hyperbolic manifolds) to determine the slope of the opposite boundary ray, and Theorem A shows that in this case it grows uniformly with dimension.

Theorem A implies that hyperbolic manifolds in dimensions $n\geqslant 6$ are of general type (in fact $K_{\overline{X}}$ being ample is much stronger), but this need not be true in low dimensions. Indeed, any rational curve with at least three punctures or any elliptic curve with at least one puncture is hyperbolic, so every Kodaira dimension can arise in dimension 1. A famous series of examples due to Hirzebruch [Hir84] shows that there are also infinitely many smooth hyperbolic surfaces with Kodaira dimension 0 (see Example 4.1). We give an independent algebraic argument that in fact hyperbolic manifolds of dimension $n\geqslant 4$ are of general type, thereby showing that there is no higher-dimensional analog of Hirzebruch's construction:

Theorem C. *Let X be a complex hyperbolic manifold of dimension $n\geqslant 4$ whose toroidal compactification $\overline{X}$ has no orbifold points. Then X is of general type.*

Thus, $K_{\overline{X}}$ is big; $K_{\overline{X}}$ is also nef for $n\geqslant 3$ by a recent theorem of Di Cerbo–Di Cerbo [DCDC] (see Theorem 4.5 below). These two facts together imply an interesting consequence to the birational geometry of such varieties: by the basepoint-free theorem [KM98, Theorem 3.3], $K_{\overline{X}}$ is in fact semi-ample. In general, for any smooth projective variety Y for which K_Y is nef, the abundance conjecture asserts that some multiple of K_Y is basepoint-free. The abundance conjecture is known in dimension $\leqslant 3$, so we obtain:

Corollary D. *Smooth toroidal compactifications $\overline{X}$ of complex hyperbolic manifolds satisfy the abundance conjecture in all dimensions.*

It is an interesting question whether Corollary B and Theorem C are sharp:

Question. *Do there exist complex hyperbolic manifolds X of non-maximal Kodaira dimension in dimension $n=3$ or for which $K_{\overline{X}}$ is not ample in dimension $n=3,4,5$?*

The behavior in Theorems A and C is common among locally symmetric varieties. For example, the moduli space of principally polarized g-dimensional abelian varieties $\mathcal{A}_g$ is known to be of general type for $g\geqslant 7$, and $K_{\overline{\mathcal{A}_g}}+(1-t)D$ is ample for $t\in(0,\frac{n+1}{12})$ on the perfect cone compactification $\overline{\mathcal{A}}_g$ by a result of Shepherd-Barron [SB06]. There are two main methods of proving positivity properties of K in this context: (a) by producing effective divisors of large slope moduli-theoretically; or (b) by using modular forms to construct sections. The novelty of our approach is that it relies only on the metric geometry of the uniformizing group, and therefore does not require either a moduli interpretation or an arithmetic lattice.

There are a number of applications of Theorems A and C. We obtain an improvement on Parker's bound [Par98] on the number of cusps of a complex hyperbolic manifold of fixed volume:

COROLLARY E. *If k is the number of cusps of X, then*

$$\frac{\mathrm{vol}(X)}{k} \geqslant \frac{2^n}{n}$$

Theorem C gives a slightly better bound in low dimensions, see Corollary 5.2 (this is also observed in [DCDC]). The bound of Corollary E is in fact equal to Parker's bound for uniformizing groups whose parabolic subgroups are unipotent [Par98, Theorem 3.1], though Corollary E applies to a larger class of lattices (see also the discussion after Corollary 5.2). This is interesting because Parker's method cannot give the same bound in this case. The main error in Parker's general result comes from bounding the minimal index of a Heisenberg lattice in the stabilizer of a cusp, which does not appear here. Cusp bounds are treated from a perspective closer to ours in [Hwa04].

With Theorem C in place, we can ask if $\overline{X}$ satisfies the Green–Griffiths conjecture:

CONJECTURE. *(Green–Griffiths [GG80]) Let Y be a smooth projective variety over $\mathbb{C}$ of general type. Then there exists a subvariety $Z \subsetneq Y$ such that every entire holomorphic map $\mathbb{C} \to Y$ factors through Z.*

The smallest such Z is called the exceptional locus. By a theorem of Nadel [Nad89], it is not difficult to show that some finite cover of $\overline{X}$ satisfies the conjecture; our main theorem Theorems A and C allow us to improve the bounds on the ramification required in such a cover:

COROLLARY F. *With X as in Theorem A, let $X' \to X$ be a finite étale cover that ramifies at each boundary component to order ℓ. Then $\overline{X}'$ satisfies the Green–Griffiths conjecture with the boundary as exceptional locus if:*

(i) *$\ell \geqslant 4$ and $n \geqslant 6$;*
(ii) *$\ell \geqslant 3$ and $n = 4, 5$.*

We can also give an intrinsic criterion that does not require passage to a cover:

COROLLARY G. *With X as in Theorem A, $\overline{X}$ satisfies the Green–Griffiths conjecture with the boundary as exceptional locus if the cusps have uniform depth greater than 2π.*

See Definition 3.7 for the notion of uniform depth. Finally, Theorem A substantially improves a variety of results about complex hyperbolic manifolds that have been proven recently using the algebraic geometry of toroidal compactifications. These methods use as input the positivity of divisors of the form $K_{\overline{X}} + (1-\lambda)D$; for $\lambda = 0$ it comes for free on any toroidal compactification. Di Cerbo–Di Cerbo [DCDC15] have systematically studied effectivity results that follow from this positivity in the range $\lambda \in [0, 2/3]$ (or more recently for $\lambda \in [0, 1]$ in [DCDC]), including bounds on the number, degree, and Picard rank of hyperbolic manifolds of a given volume. For most of these results, simply plugging Theorem A into their argument yields a better bound, and we choose to leave these modifications to the reader.

Outline

In Section 2 we collect some background on hyperbolic manifolds and their toroidal compactifications. In Section 3 we prove the volume bounds on the multiplicity of curves along the boundary, and use it to conclude Theorem A. These bounds are the boundary analogs of those proven by Hwang–To [HT02] for *interior* points of locally symmetric varieties. We provide an independent algebraic proof of Theorem C in Section 4. In Section 5 we deduce the applications in Corollaries E, F, and G.

Acknowledgements

The first named author would like to thank M. Stover, F. Gounelas, and C. Xu for useful conversations, and G. Di Cerbo in particular for introducing the authors to some of the open problems in the field and for many enlightening discussions. This paper was written during the first named author's visit to Columbia University, and he is grateful for their hospitality.

2. Background

The hyperbolic n-ball is the domain

$$\mathbb{B} = \mathbb{B}^n = \{z \in \mathbb{C}^n \,||z|^2 < 1\}.$$

It has holomorphic automorphism group $\mathrm{PU}(n,1)$ and Bergman metric

$$h = ds^2_{\mathbb{B}} = 4 \cdot \frac{(1-|z|^2)\sum_i dz_i \otimes d\overline{z}_i + (\sum_i \overline{z}_i dz_i) \otimes (\sum_i z_i d\overline{z}_i)}{(1-|z|^2)^2}$$

of constant sectional curvature -1. With this normalization, $\mathrm{Ric}(h) = -\frac{n+1}{2}h$, and the associated Kähler form is $\omega_{\mathbb{B}} = \frac{1}{2}\operatorname{Im} ds^2_{\mathbb{B}}$.

Let $\Gamma \subset \mathrm{PU}(n,1)$ be a cofinite-volume discrete subgroup and $X = \mathbb{B}/\Gamma$. X naturally has the structure of an orbifold; every elliptic element of Γ is torsion, so if Γ is torsion-free X is a smooth complex manifold. Γ always admits a finite index torsion-free (in fact neat) subgroup, by [AMRT75] in the arithmetic case and [Hum98] in general. Henceforth we will typically only consider Γ torsion-free, and we will refer to such X as torsion-free ball quotients.

The cusps of X are in one-to-one correspondence with the equivalence classes of parabolic fixed points of Γ, and the Baily–Borel compactification X^* is a normal projective variety obtained by adding one point for each cusp ([BB66] in the arithmetic case; [SY82] in general). X also admits a unique orbifold toroidal compactification $\overline{X}$ by [AMRT75] in the case of an arithmetic lattices Γ and by [Mok12] in general. If $\overline{X}$ has no orbifold points (see Definition 2.3), then it is a smooth projective variety and each connected component E of the boundary divisor D is an étale quotient of an abelian variety whose normal bundle $\mathcal{O}_E(E)$ is anti-ample. If the parabolic subgroups of Γ are unipotent (in particular if Γ is neat), the boundary D is a disjoint union of abelian varieties. In any case, the log-canonical divisor $K_{\overline{X}} + D$ is semi-ample and induces a birational map $\overline{X} \to X^*$ which is an isomorphism on the open part X and contracts each boundary component E to the point of X^* compactifying the corresponding cusp.

The hermitian metric $ds^2_{\mathbb{B}}$ descends to X and extends to a "good" singular hermitian metric on the log-tangent bundle $T_{\overline{X}}(-\log D)$ by a theorem of Mumford [Mum77]. Likewise, there is a natural singular hermitian metric on the log-canonical bundle $\omega_{\overline{X}}(D)$, and integration against the Kähler form ω_X on the open part represents (as a current) a multiple of the first Chern class dictated by our choice of normalization:

$$\mathrm{c}_1(K_{\overline{X}}+D)=\frac{1}{2\pi}\frac{n+1}{2}[\omega_X]\in H^{1,1}(\overline{X},\mathbb{R}). \tag{1}$$

For analyzing the boundary behavior in more detail, the Siegel model is more convenient. Our presentation is taken from Parker [Par98]. Let

$$\mathbb{S}=\mathbb{S}^n=\mathbb{C}^{n-1}\times\mathbb{R}\times\mathbb{R}_{>0}$$

where $\mathbb{C}^{n-1}$ is endowed with the standard positive definite hermitian form[1] $(\cdot,\cdot)$. We use coordinates (ζ,v,u), and note that holomorphic coordinates in this model are given by ζ and

$$z=v+i(|\zeta|^2+u)$$

whence

$$\mathbb{S}=\{(\zeta,z)\in\mathbb{C}^{n-1}\times\mathbb{C}\mid \operatorname{Im} z>|\zeta|^2\}.$$

The Siegel model comes with a preferred cusp at infinity whose parabolic stabilizer G_∞ contains the group of Heisenberg isometries $U_\infty:=\mathrm{U}(n-1)\ltimes\mathfrak{N}$ acting only on the first two coordinates $\mathbb{C}^{n-1}\times\mathbb{R}$: Heisenberg rotations $\mathrm{U}(n-1)$ act on $\mathbb{C}^{n-1}$ in the usual way and Heisenberg translations $\mathfrak{N}\cong\mathbb{C}^{n-1}\times\mathbb{R}$ act via

$$(\tau,t):(\zeta,v)\mapsto(\zeta+\tau,v+t+2\operatorname{Im}(\tau,\zeta)).$$

For completeness, we note that in the holomorphic coordinates this is:

$$(\tau,t):(\zeta,z)\mapsto\left(\zeta+\tau,z+t+i|\tau|^2+2i(\zeta,\tau)\right).$$

We denote by (A,τ,t) the transformation which first rotates by $A\in U(n-1)$ and then translates by (τ,t). $\mathfrak{N}$ is a central extension of the group $\mathbb{C}^{n-1}$ of translations on the first coordinate by the group $\mathbb{R}$ of translations in the second coordinate. We call translations of the form $(0,t)$ *vertical* translations, and note that the subgroup $T_\infty\subset G_\infty$ of vertical translations is the center. The group U_∞/T_∞ is identified with the group of affine unitary transformations of $\mathbb{C}^{n-1}$ via projection to the ζ coordinate.

The subgroup $U_\infty\subset G_\infty$ can be thought of as the stabilizer of the *height* coordinate u, and $-2\log u$ is a potential for the Kähler form:

Lemma 2.1. $\omega_{\mathbb{S}}=-2i\partial\overline{\partial}\log u$.

Proof. This follows from a computation and the fact that in the Siegel model the hermitian metric is

$$ds^2_{\mathbb{S}}=\frac{du^2+(dv-2\operatorname{Im}(\zeta,d\zeta))^2+4u(d\zeta,d\zeta)}{u^2}$$

(see *e.g.* [Par98]). □

The horoball $B(u)$ of height u centered at the cusp at infinity is defined to be the set

$$B(u)=\mathbb{C}^{n-1}\times\mathbb{R}\times(u,\infty).$$

It is clearly preserved by U_∞. The remaining generator of G_∞ is a one-dimensional torus which scales $(\zeta,v,u)\mapsto(a\zeta,a^2v,a^2u)$, and this scales the horoball of height u in the obvious way.

Now suppose Γ has a parabolic fixed point at infinity, and let $\Gamma_\infty=\Gamma\cap G_\infty$ be its stabilizer. For any horoball $B(u)$ centered at infinity, define the horoball neighborhood $V(u):=B(u)/\Gamma_\infty$. Note that at some sufficiently large height u, $V(u)$ injects into X by Shimizu's lemma.

[1] Our hermitian forms are $\mathbb{C}$-linear in the first variable.

Definition 2.2. We call the smallest u such that $V(u)$ injects the height u_∞ of the cusp, and we call $s_\infty = t_\infty/u_\infty$ the depth. Note that the depth of a cusp is invariant under conjugating the lattice Γ, whereas the height is not.

The partial quotient by the vertical translations $\Theta_\infty = \Gamma \cap T_\infty$ is given by the map

$$\mathbb{S} \to \mathbb{C}^{n-1} \times \Delta^* : (\zeta, z) \mapsto (\zeta, e^{2\pi i z/t_\infty}).$$

The cusp is compactified by taking the interior closure in $\mathbb{C}^{n-1} \times \Delta$, which corresponds to adding a boundary component of the form $D_\infty = \mathbb{C}^{n-1} \times 0/\Lambda_\infty$, where $\Lambda_\infty := \Gamma_\infty/\Theta_\infty$ is identified with a discrete group of affine unitary transformations. At every point $\zeta \in \mathbb{C}^{n-1} \times 0$ there is a character $\chi_\infty(\zeta)$ encoding the action on q. Note that $\chi_\infty(\zeta)$ has finite order since Γ is discrete, and we call its order $m_\infty(\zeta)$. We define $m_\infty = \max_\zeta m_\infty(\zeta)$ to be the order of the cusp.

Definition 2.3. We say that Γ is *torsion-free at infinity* if Γ is torsion-free and Λ_∞ is torsion-free for each parabolic fixed point q_∞. Equivalently, Γ is torsion-free at infinity if the orbifold toroidal compactification $\overline{X}$ of $X = \mathbb{B}/\Gamma$ has no orbifold points.

Note that the condition that $\overline{X}$ have smooth coarse space is slightly weaker, as we only need every parabolic element to have one non-identity eigenvalue. Neat groups are clearly torsion-free at infinity, as are groups all of whose parabolic subgroups are unipotent. Every Γ contains a finite-index neat subgroup, so clearly every complex hyperbolic orbifold X has a finite étale cover X' whose uniformizing group is torsion-free at infinity.

If Γ is torsion-free at infinity, then the residual quotient of $\mathbb{C}^{n-1} \times \Delta$ by Λ_∞ is étale, so locally around the boundary we have coordinates $\zeta, q = e^{2\pi i z/t_\infty}$ and the boundary is cut out by $q = 0$. $V(u)$ is then identified with a neighborhood of the zero section in the normal bundle $\mathcal{O}_{D_\infty}(D_\infty)$ (see [Mok12]). In general, $q^{m_\infty(\zeta)}$ locally descends to a function on the coarse space of $\overline{X}$ which vanishes along the boundary.

3. Boundary multiplicity inequalities and ampleness

Let $X = \mathbb{B}/\Gamma$ be a torsion-free ball quotient and suppose q_∞ is a parabolic fixed point of Γ with stabilizer $\Gamma_\infty = \Gamma \cap G_\infty$. By considering the Siegel model associated to q_∞, we have by the previous section horoball neighborhoods $V(u) \subset X$ for all $u < u_\infty$, where u_∞ is the height of q_∞. Let $\overline{V}(u)$ be the interior closure of $V(u)$ in the toroidal compactification $\overline{X}$.

We first show that the volume of an analytic subvariety of the horoball neighborhood $\overline{V}(u)$ scales as the height of the horoball drops.

Proposition 3.1. *Let Y be an irreducible k-dimensional analytic subvariety of $\overline{V}(u)$ not contained in the boundary. Then*

$$u^k \operatorname{vol}(Y \cap V(u))$$

is a non-increasing function of $u > u_\infty$.

Of course, Proposition 3.1 is equally true in the orbifold setting, since we may simply pass to a torsion-free cover.

Before the proof we recall a lemma of Demailly [Dem12]. Let X be a complex manifold and $\varphi : X \to [-\infty, \infty)$ a continuous plurisubharmonic function. Define

$$B_\varphi(r) = \{x \in X \mid \varphi(x) < r\}.$$

We say φ is semi-exhaustive if the balls $B_\varphi(r)$ have compact closure in X. Further, for T a closed positive current of type (p,p), we say φ is semi-exhaustive on $\operatorname{Supp} T$ if the same is true for $B_\varphi(r) \cap \operatorname{Supp} T$. In this case, the integral

$$\int_{B_\varphi(r)} T \wedge (i\partial\overline{\partial}\varphi)^p := \int_{B_\varphi(r)} T \wedge (i\partial\overline{\partial}\max(\varphi, s))^p$$

is well-defined and independent of $s < r$ [Dem12, §III.5] (see also [HT02]). We then have the following:

LEMMA 3.2. *(Formula III.5.5 of [Dem12]) For any convex increasing function $f : \mathbb{R} \to \mathbb{R}$,*

$$\int_{B_\varphi(r)} T \wedge (i\partial\overline{\partial} f \circ \varphi)^p = f'(r-0)^p \int_{B_\varphi(r)} T \wedge (i\partial\overline{\partial}\varphi)^p$$

where $f'(r-0)$ is the derivative of f from the left at r.

Proof of Proposition 3.1.

$$\begin{aligned}
\operatorname{vol}(Y \cap V(u_0)) &= \frac{1}{k!}\int_{Y\cap V(u_0)} \omega_X^k \\
&= \frac{1}{k!}\int_{Y\cap V(u_0)} (i\partial\overline{\partial}(-2\log u))^k \\
&= \frac{1}{k!}\int_{V(u_0)} (i\partial\overline{\partial}(-2\log u))^k \wedge [Y] \\
&= \frac{2^k u_0^{-k}}{k!}\int_{V(u_0)} (i\partial\overline{\partial}(-u))^k \wedge [Y] \\
&= \frac{2^k u_0^{-k}}{k!}\int_{Y\cap V(u_0)} (i\partial\overline{\partial}(-u))^k.
\end{aligned}$$

As $-u$ is plurisubharmonic,

$$u_0^k \operatorname{vol}(Y \cap V(u_0)) = \frac{2^k}{k!}\int_{Y\cap V(u_0)} (i\partial\overline{\partial}(-u))^k$$

is a non-increasing function of u_0 (the horoballs $V(u_0)$ *shrink* as u_0 grows). □

Taking the limit of Proposition 3.1 as $u \to 0$ yields a bound on the multiplicity of a curve at the boundary in terms of its volume in a horoball neighborhood.

PROPOSITION 3.3. *Assume $\Lambda_\infty = \Gamma_\infty/\Theta_\infty$ is torsion-free and let t_∞ be the length of the smallest vertical translation $(0, t_\infty) \in \Gamma_\infty$. For any irreducible 1-dimensional analytic subvariety C of $\overline{V}(u)$ not contained in the boundary and any $u > u_\infty$, we have*

$$\operatorname{vol}(C \cap V(u)) \geqslant \frac{t_\infty}{u} \cdot (C \cdot D_\infty)$$

where D_∞ is the divisor compactifying q_∞ in the toroidal compactification.

Proof. From the proof of the previous proposition, we just need to compute

$$2 \cdot \lim_{u_0\to\infty} \int_{C\cap V(u_0)} i\partial\overline{\partial}(-u).$$

For u_0 sufficiently large, $C \cap V(u_0)$ is a union of pure 1-dimensional analytic sets, each component of which is normalized by a disk $f_j : \Delta_j \to C \cap V(u_0)$. We may assume $f_j(0) = x_j \in D_\infty$ and that $f_j|_{\Delta_j^*}$ is an isomorphism onto an open set of C. $q = e^{2\pi i z/t_\infty}$ is a local defining equation for D_∞ and we have

$$C \cdot D_\infty = \sum_j \operatorname{ord} f_j^* q.$$

Now for sufficiently large u_0, we have

$$\int_{C\cap V(u_0)} i\partial\overline{\partial}(-u) = \sum_j \int_{\Delta_j} f_j^* \partial\overline{\partial}(-u)$$

but of course

$$\int_{\Delta_j} f_j^* \partial\overline{\partial}(-u) \geqslant \pi \cdot \nu(f_j^*(-u), 0).$$

If t is a uniformizer for Δ_j at 0, then we compute

$$\begin{aligned}
\nu(f_j^*(-u), 0) &= \liminf_{t\to 0} \frac{f_j^*(-u)}{\log|t|} \\
&= \liminf_{t\to 0} \frac{1}{\log|t|} \cdot f_j^* \left(|\zeta|^2 + \frac{t_\infty}{2\pi} \cdot \log|q| \right) \\
&= \frac{t_\infty}{2\pi} \cdot \operatorname{ord} f_j^* q.
\end{aligned}$$

□

Remark 3.4. If we don't assume Λ_∞ is torsion-free, then we've proven

$$\operatorname{vol}(C \cap V(u)) \geqslant \frac{t_\infty}{u} \cdot (\widetilde{C \cdot D_\infty})$$

where we've defined a weighted intersection product

$$(\widetilde{C \cdot D_\infty}) = \sum_{x \in D_\infty} \frac{1}{m_\infty(x)} (C \cdot D_\infty)_x.$$

Here $(C \cdot D_\infty)_x$ is the contribution of x to the usual intersection product on the coarse space of $\overline{X}$, and $m_\infty(x)$ is the order of x defined at the end of Section 2.

Remark 3.5. Proposition 3.3 is sharp in the sense that a union of vertical complex geodesics will realize the equality. A vertical complex geodesic is a copy of the upper half-plane $\mathbb{H} \subset \mathbb{S}$ embedded as $\zeta = 0$ (or a horizontal translate thereof), and the intersection of $\mathbb{H}$ with the horoball $B(u)$ is $\mathbb{H}_{>u} = \{z \in \mathbb{H} \mid \operatorname{Im} z > u\}$. The resulting curve C in $V(u)$ is the quotient of $\mathbb{H}_{>u}$ by real translation by t_∞ and therefore has $\operatorname{vol}(C \cap V(u)) = t_\infty/u$. Finally, we have $(C \cdot D_\infty) = 1$, as C is uniformized by $0 \times \Delta$ in the partial quotient $\mathbb{C}^{n-1} \times \Delta$.

Proposition 3.3 is analogous to the multiplicity bound proven by Hwang–To [HT02] for an *interior* point x of a quotient of a bounded symmetric domain. They show for a k-dimensional subvariety that

$$\operatorname{vol}(Y \cap B(x, r)) \geqslant \operatorname{vol}(D(r))^k \cdot \operatorname{mult}_x Y$$

where $B(x, r)$ is an isometrically embedded hyperbolic ball around x of radius r and $D(r)$ is the volume in $B(x, r)$ of a complex geodesic through x. One can show in this case a relative version as in Proposition 3.1 as well.

We could have proven Proposition 3.3 directly by methods more analogous to [HT02]. As in Section 2, the hermitian metric h on ω_X extends to a singular hermitian metric $\overline{h}$ on $\omega_{\overline{X}}(D)$. We form a different singular metric by twisting by a function $e^{-\varphi}$ supported on $V(u_0)$ so that $e^{-\varphi}\overline{h}$ has positive curvature form and Lelong number $\frac{t_\infty}{u_0}$ at every point of the boundary. As $\overline{h}$ is given by $e^{2\log u}$ on $V(u)$, taking φ so that $\varphi - 2\log u$ approximates the tangent line to $-2\log u$ at u_0 will achieve this. We choose instead to derive Proposition 3.3 from Proposition 3.1 because the latter statement is interesting (and useful, *e.g.* [BT]) in its own right.

We are now in a position to prove Theorem A, which will follow from Corollary 3.9 below. Let q_i be the cusps of X, and denote by D_i the boundary component of $\overline{X}$ compactifying q_i. Let t_i be the length of the smallest vertical translation in the stabilizer of q_i.

Now suppose for each cusp q_i we choose a horoball height u_i such that:

($*$) each $V(u_i)$ injects into X (*i.e.* u_i is less than the height of q_i);

($**$) the $V(u_i)$ are all disjoint.

Proposition 3.6. *Let Γ be torsion-free at infinity and $\overline{X}$ the toroidal compactification of $X = \mathbb{B}/\Gamma$ with boundary D. Let $L = K_{\overline{X}} + D$. Then in the above situation,*

$$L - \frac{n+1}{4\pi}\sum_i s_i D_i \tag{2}$$

is ample for $s_i \in \left(0, \frac{t_i}{u_i}\right)$.

Proof. By (1) and Proposition 3.3, the divisor is nef modulo the boundary, but for any component E of the boundary $K_{\overline{X}}|_E \equiv -E|_E$ is ample so it is in fact nef. Moreover, $L - \epsilon D$ is ample for all sufficiently small $\epsilon > 0$ (see [DCDC15]). As the interior of any line drawn between a point of the nef cone and a point in the ample cone is contained in the ample cone, the claim follows. □

For convenience, we make the following

Definition 3.7. The uniform depth s of the cusps of X is the largest $s > 0$ such that, setting $u_i = t_i/s$, the horoball neighborhoods $V(u_i)$ satisfy properties ($*$) and ($**$).

Corollary 3.8. *In the above setup, if the cusps of X have uniform depth s, then $L - \lambda D$ is ample for $0 < \lambda < \frac{n+1}{2\pi}s$.*

Corollary 3.9. *In the above setup, $L - \lambda D$ is ample for $0 < \lambda < \frac{n+1}{2\pi}$. In particular, $K_{\overline{X}}$ is ample if $n \geqslant 6$.*

Proof. By [Par98, Proposition 2.4], since Γ is torsion-free the cusps have uniform depth at least 2. The second claim follows since $\frac{n+1}{2\pi} > 1$ for $n \geqslant 6$. □

Remark 3.10. If we only assume Γ is cofinite-volume (and require no torsion-freeness), then the above proof still goes through, and by Remark 3.4 we conclude that (2) is ample on the coarse space $\overline{X}$ provided $s_i \in \left(0, \frac{t_i}{m_i u_i}\right)$, where m_i is the order of the cusp compactified by D_i, as defined in the previous section. On the *orbifold* compactification Proposition 3.3 and Proposition 3.6 are true as written, since intersection numbers, ampleness, and nefness can computed/checked on a finite cover with trivial stabilizers. We leave the details to the reader.

4. Kodaira dimension

In this section we prove Theorem C. Theorem A is stronger starting in dimension $n \geqslant 6$, so this section is strictly speaking only necessary to handle the case of fourfolds and fivefolds. On the other hand, our proof is independent of the results of Section 3 and entirely algebraic.

We begin with an example due to Hirzebruch [Hir84] for context.

Example 4.1. Let $\zeta = e^{2\pi i/3}$ and $E = \mathbb{C}/\mathbb{Z}[\zeta]$ be the elliptic curve with $j = 0$. Consider the blow-up S of $E\times E$ at the origin $0 \in E\times E$. We have $K_S \equiv F$ where F is the exceptional divisor. If we let D be the union of the strict transforms of the fibers $E \times 0$, $0 \times E$, and the graphs of $1, -\zeta \in \mathbb{Z}[\zeta]$, then the complement $U = S \smallsetminus D$ is uniformized by $\mathbb{B}^2$ by a theorem of Yau, since we compute

$$3 = (F + D)^2 = \mathrm{c}_1\left(\omega_S(\log D)\right)^2 = 3\,\mathrm{c}_2\left(\Omega^1_S(\log D)\right) = 3\chi(U)$$

and $K_S + D$ is big and nef. It follows that S is the toroidal compactification of U with boundary D.

Hirzebruch's example shows that the toroidal compactification of a torsion-free (in fact neat) ball quotient in dimension 2 may be non-minimal (*i.e.* $K_{\overline{X}}$ is not nef) and may have Kodaira dimension 0. Blow-ups of $E\times E$ at special configurations of points for other elliptic curves E yield infinitely many such examples. Recently, Di Cerbo and Stover [DCS] have given some examples birational to bielliptic surfaces.

The main goal of this section is to show that neither of these phenomena can occur in higher dimensions, and in particular that every complex hyperbolic manifold of dimension $\geqslant 4$ is of general type. Recall that a quasiprojective variety X is of general type if some projective compactification X' has maximal Kodaira dimension, $\kappa(X') = n$.

Proposition 4.2. *Let Γ be torsion-free at infinity and $X = \mathbb{B}/\Gamma$. If $n \geqslant 4$, then $\overline{X}$ is of general type.*

In fact, $K_{\overline{X}}$ is big and nef: the nefness is a recent result of Di Cerbo–Di Cerbo [DCDC], and holds in dimension $n \geqslant 3$. Recall that the abundance conjecture asserts that for a smooth projective variety Y with K_Y nef, then in fact K_Y is semi-ample, and is known in dimension $\leqslant 3$ (see for example [Kaw92] for the resolution of the final case). By the basepoint-free theorem [KM98, Theorem 3.3], we can conclude that this is the case for toroidal compactifications of complex hyperbolic manifolds:

Corollary 4.3. *With X as above, $K_{\overline{X}}$ is semi-ample,* i.e. *$\overline{X}$ satisfies the abundance conjecture.*

Our proof of Propostion 4.2 will only require the coarse space of $\overline{X}$ to be smooth up until the last step in Lemma 4.10. For completeness, we first summarize the argument of [DCDC] for the nefness of $K_{\overline{X}}$ using the cone theorem and bend-and-break.

Given a smooth curve C, a projective variety Y, a set of points $S \subset C$, and a map $f|_S : S \to Y$, we denote by $\mathrm{Hom}(C, Y; f|_S)$ the scheme parametrizing maps $f : C \to Y$ restricting to $f|_S$ along S. Recall that bend-and-break says (see [Deb01, Propositions 3.1 and 3.2]) the following:

Proposition 4.4 Bend-and-break. *Let Y be a projective variety. For any map $f : \mathbb{P}^1 \to X$ and any (quasiprojective) curve $B \subset \mathrm{Hom}(\mathbb{P}^1, X; f|_{\{0,\infty\}})$ containing f along which $f_b(\mathbb{P}^1)$ is not constant, $f_b(\mathbb{P}^1)$ has a limit which is a reducible or multiple rational curve.*

The key idea for us is that an extremal $K_{\overline{X}}$-negative rational curve $f:\mathbb{P}^1\to\overline{X}$ must intersect the boundary D in at least 3 points since X is uniformized by a bounded domain. On the other hand, $f:\mathbb{P}^1\to\overline{X}$ deforms, since for any component B of $\mathrm{Hom}(\mathbb{P}^1,\overline{X})$ containing f,

$$\dim B \geqslant -K_{\overline{X}}\cdot f(\mathbb{P}^1)+\dim\overline{X}. \tag{3}$$

As long as $n\geqslant 3$, then $\dim B\geqslant 4$, and in the Baily–Borel compactification X^* we have a family of rational curves with 3 fixed points, so by bend-and-break $f(\mathbb{P}^1)$ is algebraically equivalent to a reducible or multiple rational curve. Note that each component of the boundary is an étale quotient of an abelian variety and therefore has no rational curves. The log canonical bundle $L=K_{\overline{X}}+D$ is big and nef and induces the map to X^*, so by induction on the degree with respect to $L=K_{\overline{X}}+D$ we have a contradiction. Thus, $\overline{X}$ can only be non-minimal if $n=2$:

Theorem 4.5. *(Theorem 1.1 of [DCDC]) $K_{\overline{X}}$ is nef if $n\geqslant 3$.*

Remark 4.6. Given Theorem 4.5, to prove Proposition 4.2 it would be enough to show that

$$K_{\overline{X}}^n=L^n+(-D)^n>0.$$

For any component E of the boundary, $-(-E)^n$ computes the rate of growth of the volume of a horoball neighborhood of E, and L^n computes the global volume of X, up to a normalization. The best known bounds on the size of distinct horoball neighborhoods give bigness for $n\geqslant 6$ (but only in the case of neat quotients); one could conceivably finish the proof of Proposition 4.2 by a case by case analysis as in Parker [Par98]. We instead pursue the algebraic line of attack.

We now need to understand curves C for which $K_{\overline{X}}.C=0$; we call such curves $K_{\overline{X}}$-trivial. We call an (irreducible) curve C rigid if no component of $\mathrm{Hom}(\widetilde{C},\overline{X})$ containing the normalization $\widetilde{C}\to\overline{X}$ has dimension greater than the dimension of the infinitesimal automorphism group $\dim H^0(\widetilde{C},T_{\widetilde{C}})$ (that is, $3,1,0$ for $g(\widetilde{C})=0,1,\geqslant 2$ respectively).

Lemma 4.7. *For $n\geqslant 4$, there are no $K_{\overline{X}}$-trivial rational curves.*

Proof. If $f:\mathbb{P}^1\to\overline{X}$ has $K_{\overline{X}}\cdot f(\mathbb{P}^1)=0$, then for any component B of $\mathrm{Hom}(\mathbb{P}^1,\overline{X})$ containing f for which $\dim B\geqslant 4$, $f(\mathbb{P}^1)$ is algebraically equivalent to a sum $\sum_i C_i$ of integral rational curves by bend-and-break. Since $K_{\overline{X}}$ is nef, we have $K_{\overline{X}}\cdot C_i=0$ for each i. By induction on the degree with respect to an ample bundle, we can repeat the same argument for each C_i and thus we may assume the C_i are all rigid. By (3), if $n\geqslant 4$ there are no rigid $K_{\overline{X}}$-trivial rational curves, a contradiction. □

Note that Lemma 4.7 fails in dimension 3 because (3) does not rule out the existence of rigid rational $K_{\overline{X}}$-trivial curves.

If we assume the abundance conjecture, then Lemma 4.7 is enough to conclude Proposition 4.2. Indeed, by Theorem 4.5, $K_{\overline{X}}$ would then be semi-ample, so let $f:\overline{X}\to Z$ be the fiber space induced by $|mK_{\overline{X}}|$ for $m\gg 0$. For any fiber F and any curve $C\subset F$, $K_{\overline{X}}\cdot C=0$ whereas $D|_D\equiv -K_{\overline{X}}|_D$ is anti-ample. We must therefore have $\dim(D\cap F)=0$, so $\dim F=n-\kappa(\overline{X})\leqslant 1$. But if $\kappa(\overline{X})=n-1$, then the general fiber F is a $K_{\overline{X}}$-trivial elliptic curve. Again because X is uniformized by a bounded domain, $E\cdot F\geqslant 1$ for some component E of the boundary. Taking a curve in $C\subset E$ such that the fibers of $\mathcal{E}=f^{-1}(f(C))$ have fixed j-invariant, C is a multisection of $\mathcal{E}/f(C)$, so base-changing to C we have an isotrivial family $\mathcal{E}/C$ with a section. Projecting to X^*, this is a family of maps $F\to X^*$ fixing $0\in F$, and by bend-and-break there is a $K_{\overline{X}}$-trivial rational curve. For $n\geqslant 4$, this contradicts Lemma 4.7.

The best that is currently known is a rational version of the abundance conjecture. Recall that for M a nef line bundle on a normal projective variety Y, there is a nef reduction map $f : Y \dashrightarrow Z$ to a normal variety Z [BCE+02]. This map is the unique (up to birational equivalence on Z) dominant rational map with connected fibers such that

(i) f is "almost holomorphic" in the sense that if $U \subset Y$ is the maximal open set on which f is defined, $f : U \to Z$ has a proper fiber (and therefore the general fiber is proper);
(ii) M is numerically trivial on all proper fibers of dimension $\dim Y - \dim Z$;
(iii) For a general point $y \in Y$ and any irreducible curve C through y with $\dim f(C) = 1$ we have $M \cdot C > 0$.

We then call $n(M) := \dim Z$ the *nef dimension* of M and $n(Y) := n(K_Y)$ the *nef dimension* of Y if K_Y is nef.

LEMMA 4.8. *$\overline{X}$ has maximal nef dimension if $n \geqslant 4$.*

Proof. Take $M = K_{\overline{X}}$ on $Y = \overline{X}$ and let F be a general fiber of the nef reduction. $K_F = K_{\overline{X}}|_F$ is numerically trivial, so $D|_F$ is ample, and therefore it must again be the case that $\dim(F \cap D) = 0$. This can only happen if $\dim F \leqslant 1$. If $\dim F = 0$, we're done, while if $\dim F = 1$, F is an elliptic curve, and the argument above provides the contradiction. □

We would like to show that Lemma 4.8 implies that $K_{\overline{X}}$ is big, but in general for a nef line bundle M it is only the case that

$$n(M) \geqslant \nu(M) \geqslant \kappa(M)$$

where $\nu(M)$ is the numerical dimension and $\kappa(M)$ is the Iitaka dimension of M. If $M = K_Y$ is the canonical bundle of a smooth projective variety Y, then the abundance conjecture implies all three are equal, but we can already see that maximal nef dimension implies bigness assuming $\kappa(X)$ is sufficiently large:

LEMMA 4.9. *Let Y be a smooth n-dimensional projective variety with K_Y nef. If $n(Y) = n$ and $\kappa(Y) \geqslant n-2$, then in fact $\kappa(Y) = n$.*

Proof. Let $f : Y' \to Z$ be the Iitaka fibration of K_Y, which admits a birational morphism $g : Y' \to Y$, and let F be a very general fiber of f. We know that $\dim F = n - \kappa(Y) \leqslant 2$, that $g^*K_Y|_F$ has Iitaka dimension 0, and that $\kappa(F) = 0$ (*e.g.* [Laz04, §2.1.C]). We also know $g^*K_Y|_F$ is nef and nonzero on every curve through a very general point of F by the assumptions, which immediately implies $\dim F \neq 1$. If $\dim F = 2$, then for some effective divisor E, $g^*K_Y|_F + E = K_F$, but K_F is numerically equivalent to a sum of -1 curves since $\kappa(F) = 0$, by the Enriques–Kodaira classification of surfaces. Thus there is a curve C in F with $K_F \cdot C = 0$ and $C \cdot E \geqslant 0$ while $g^*K_Y \cdot C > 0$, which is a contradiction. □

Given Lemmas 4.8 and 4.9, the proof of Proposition 4.2 will be completed by the following:

LEMMA 4.10. *$\overline{X}$ has Kodaira dimension $\kappa(\overline{X}) \geqslant n-2$ for $n \geqslant 3$.*

Proof. Let $L = K_{\overline{X}} + D$. As $K_{\overline{X}}$ is nef by Lemma 4.5, we have

$$K_{\overline{X}}^n = (L-D)^n = L^n + (-D)^n \geqslant 0.$$

If we have strict inequality, then $K_{\overline{X}}$ is big as it is already nef by Theorem 4.5. Thus, we need only treat the case $K_{\overline{X}}^n = 0$, *i.e.* $L^n + (-D)^n = 0$.

We first note that for $t \geqslant 2$ we have $H^i(\overline{X}, tK_{\overline{X}}) = 0$ for $i > 1$, by the sequence

$$0 \to \mathcal{O}_{\overline{X}}(tK_{\overline{X}}) \to \mathcal{O}_{\overline{X}}(L + (t-1)K_{\overline{X}}) \to \mathcal{O}_D((t-1)K_{\overline{X}}) \to 0$$

and Kawamata-Viehweg vanishing applied to the second and third terms. Thus,

$$\begin{aligned} h^0(\overline{X}, tK_{\overline{X}}) &\geqslant \chi(\overline{X}, tK_{\overline{X}}) \\ &= \frac{\mathrm{c}_1(\overline{X})^2 + \mathrm{c}_2(\overline{X})}{12} \cdot \frac{(tK_{\overline{X}})^{n-2}}{(n-2)!} + O(t^{n-3}) \\ &= \frac{\mathrm{c}_2(\overline{X})}{12} \cdot \frac{(tK_{\overline{X}})^{n-2}}{(n-2)!} + O(t^{n-3}) \end{aligned}$$

by our assumption. By the sequence

$$0 \to \mathcal{O}_D(-D) \to \Omega^1_{\overline{X}}|_D \to \Omega^1_D \to 0$$

and the fact that each component of D is an étale quotient of an abelian variety, we have $\mathrm{c}_2(\overline{X}) \cdot D \equiv 0$, so by Hirzebruch proportionality [Mum77]

$$\mathrm{c}_2(\overline{X}) \cdot K_{\overline{X}}^{n-2} = \mathrm{c}_2(X) \cdot L^{n-2} = \frac{\mathrm{c}_2(\mathbb{P}^n) \cdot \mathrm{c}_1(\mathbb{P}^n)^{n-2}}{\mathrm{c}_1(\mathbb{P}^n)^n} L^n > 0$$

as $\mathbb{P}^n$ is the compact dual of $\mathbb{B}^n$. □

5. Applications

We now apply Corollary 3.9 to derive Corollaries E, F, G and some other consequences. One immediate application is a bound on the number of cusps of X:

Proposition 5.1. *For Γ torsion-free at infinity, let k be the number of cusps of $X = \mathbb{B}/\Gamma$. Then*

$$k \leqslant \frac{(2\pi)^n}{(n+1)^n} \cdot \frac{L^n}{(n-1)!}.$$

Further, in dimensions $n = 3, 4, 5$, we have

$$k \leqslant \frac{L^n}{(n-1)!}.$$

Proof. Note that each component of the boundary is an étale quotient of an abelian variety so all of the Chern classes of Ω_D vanish numerically. $D|_D$ is anti-ample, so on the one hand

$$k \leqslant \frac{D \cdot (-D)^{n-1}}{(n-1)!} = -\frac{(-D)^n}{(n-1)!} = \chi(D, \mathcal{O}_D(-D))$$

but on the other hand if $aL - bD$ is a nef $\mathbb{R}$-divisor for $a, b > 0$,

$$0 \leqslant (aL - bD)^n = a^n L^n + b^n (-D)^n.$$

Thus

$$k \leqslant \left(\frac{a}{b}\right)^n \cdot \frac{L^n}{(n-1)!}.$$

By Corollary 3.9, we can take $a = 1$ and $b = \frac{n+1}{2\pi}$. By Theorem 4.5, for $n = 3, 4, 5$ we can do better with $a = 1$ and $b = 1$. □

A similar argument is used by Di Cerbo–Di Cerbo to give an improvement to Parker's cusp bound in dimensions 2 [DCDC14] and 3 [DCDC15]. Note that by (1) we have

$$\mathrm{vol}(X) = \frac{(4\pi)^n}{n!(n+1)^n} \cdot L^n$$

and so we can restate the best known bounds in this context:

Corollary 5.2. *Let k be the number of cusps of X. Then*

$$\frac{\mathrm{vol}(X)}{k} \geqslant \begin{cases} \frac{\pi^2}{2} & n = 2 \\ \frac{(4\pi)^n}{n(n+1)^n} & n = 3,4,5 \\ \frac{2^n}{n} & n \geqslant 6. \end{cases}$$

The bound of Corollary 5.2 in dimension $n = 2$ is sharp and due to Di Cerbo–Di Cerbo [DCDC14]. For $n \geqslant 6$, the above bound is equal to that derived by Parker [Par98] in the case that the parabolic subgroups of Γ are unipotent; we show that the same bound holds for the larger class of Γ torsion-free at infinity. On the other hand, the argument of Proposition 5.1 could conceivably improve Parker's bound for all torsion-free Γ if m_∞ can be controlled sufficiently well.

The proofs of Corollaries F and G will follow from a result of Nadel [Nad89, Theorem 2.1]: if Y is a finite-volume quotient of a bounded symmetric domain whose holomorphic sectional curvature is $\leqslant -\gamma$ (with the normalization $\mathrm{Ric}(h) = -h$) for some $\gamma \in \mathbb{Q}$, then for any smooth toroidal compactification $\overline{Y}$ such that $K_{\overline{Y}} + (1 - 1/\gamma)D$ is big, every entire map $\mathbb{C} \to \overline{Y}$ has image contained in the boundary.

Proof of Corollaries F and G. For us, $\gamma = \frac{2}{n+1}$, and Corollary G is immediate from Corollary 3.8.

For the first part of Corollary F, if $f : X' \to X$ is a cover ramifying to order at least ℓ along each boundary component, then $f^*D \geqslant \ell D'$. We have

$$\begin{aligned} f^*\left((K_{\overline{X}} + D) - tD\right) &= (K_{\overline{X}'} + D') - t(f^*D') \\ &\leqslant (K_{\overline{X}'} + D') - t\ell D'. \end{aligned}$$

In any dimension, we can then take $t = \frac{n+1}{8}$ and the left hand side is big by Corollary 3.9, so $\ell = 4$ is sufficient. This can be slightly improved in dimension $n = 4, 5$ since the same is true for $t = 1$ by Proposition 4.2, and now $\ell = 3$ will do. $\square$

The following corollary is a well-known consequence of Nadel's theorem for arithmetic quotients but in fact the same proof holds for non-arithmetic quotients given the work of [Mok12]. We include it for completeness, but the main point of Corollary F is the improved control over the ramification order.

Corollary 5.3. *Every complex hyperbolic orbifold X admits a finite étale cover X' such that the toroidal compactification $\overline{X}'$ satisfies the Green–Griffiths conjecture with the boundary as exceptional locus.*

Of course, this is equivalent to the Baily–Borel compactification X'^* having no nontrivial entire maps $\mathbb{C} \to X'^*$.

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