A SHORT PROOF OF A CONJECTURE OF MATSUSHITA

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Abstract. We build on the arguments of van Geemen and Voisin [24] to prove a conjecture of Matsushita that a Lagrangian fibration of an irreducible hyperkähler manifold is either isotrivial or of maximal variation. We also complete a partial result of Voisin [26] regarding the density of torsion points of sections of Lagrangian fibrations.

Let X be an irreducible compact hyperkähler manifold, that is, a simply-connected compact Kähler manifold X for which $H^0(X,\Omega^2_X) = \mathbb{C}\sigma$ for a nowhere-degenerate holomorphic two-form σ. A Lagrangian fibration of X is a proper morphism $f : X \to B$ to a normal compact analytic variety B whose generic fiber is smooth, connected, and Lagrangian (see [14] for a recent survey). By a result of Voisin (see for example [6]), it follows that every smooth fiber is an abelian variety. We let $B^\circ \subset B$ be a dense Zariski open smooth subset over which the base-change $f^\circ : X^\circ \to B^\circ$ is smooth. By the period map of f we mean the period map $\varphi : B^\circ \to S$ to an appropriate moduli space S of polarized abelian varieties associated to the natural variation of (polarized) weight one integral Hodge structures on B° with underlying local system $R^1f^\circ_*\mathbb{Z}_{X^\circ}$. We say f is *isotrivial* if the period map is trivial (equivalently if $R^1f^\circ_*\mathbb{Z}_{X^\circ}$ has finite monodromy) and *of maximal variation* if the period map is generically finite.

Our main result is to resolve a conjecture of Matsushita:

Theorem 1. *Let X be an irreducible hyperkähler manifold (or more generally a $\mathbb{Q}$-factorial terminal primitive symplectic variety in the sense of* [2]*). Then any Lagrangian fibration $f : X \to B$ is either isotrivial or of maximal variation.*

Both possibilities in Theorem 1 occur for K3 surfaces S—see for example [13, Chapter 11]—and therefore also for their Hilbert schemes $S^{[g]}$ in each (even) dimension. Primitive symplectic varieties are the natural singular analog (as far as deformation theory is concerned) of irreducible hyperkähler manifolds; see below for the definition and the precise meaning of a Lagrangian fibration in this context.

Theorem 1 has previously been treated in two main contexts. First, the Beauville–Mukai system of an ample divisor on a K3 surface has been shown to be of maximal variation in many cases by Ciliberto–Dedieu–Sernesi [7] by studying the extendability of a canonically embedded curve to a K3 surface (where in fact the period map is shown to be quasifinite) and by Dutta–Huybrechts [10] by understanding the derivative of the period map. In particular, Dutta–Huybrechts show that Theorem 1 implies a complete answer:

Corollary 2. *Let H be a basepoint-free ample divisor on a K3 surface S. Then the complete linear system $|H|$ is of maximal variation.*

Proof. The genus 2 case is proven unconditionally in [10, Prop. 5.4], and the genus $g \geq 3$ case in [10, Prop. 5.2] assuming Theorem 1. □

Second, van Geemen and Voisin have proven Theorem 1 generically for $b_2 \geq 7$. More precisely, let $T_0 \subset H^2(X,\mathbb{Q})$ be the rational transcendental lattice, namely, the smallest rational Hodge substructure containing $[\sigma] \in H^{2,0}(X)$. Assuming that X is projective, T_0 has generic (special) Mumford–Tate group (namely $\mathrm{SO}(T_0, q_X)$, where q_X is the Beauville–Bogomolov–Fujiki form), and $\operatorname{rk} T_0 \geq 5$, van Geemen and Voisin [24, Theorem 5] show that any fiber of a Lagrangian fibration that is not of maximal variation must be a factor of the Kuga–Satake variety of T_0, hence the fibration is locally constant. Their result in particular applies to the very general projective deformation of $f : X \to B$ assuming $b_2(X) \geq 7$, which includes all known deformation types.

The argument of van Geemen–Voisin therefore eventually relies on the largeness of the generic Mumford–Tate group. We will adapt their proof to prove Theorem 1 by replacing this input with the near simplicity of the complex variation of Hodge structures on $R^1 f^\circ_* \mathbb{C}_{X^\circ}$ which holds without any genericity assumption. We first recall the basic properties of complex variations. A complex variation of Hodge structures of weight w on a smooth analytic variety (see for example [8]) consists of a $\mathbb{C}$-local system V and a holomorphic (resp. antiholomorphic) descending filtration $F^\bullet$ (resp. $F'^\bullet$) such that we have a splitting of the sheaf of C^∞ sections $A^0(V) = \bigoplus_p A^0(V^{p,w-p})$ where $V^{p,w-p} := F^p \cap F'^{w-p}$ and the flat connection maps $A^0(V^{p.w-p})$ to $A^{1,0}(V^{p-1,w-p+1}) \oplus A^1(V^{p,w-p}) \oplus A^{0,1}(V^{p+1,w-p-1})$. We refer to the p-grading of $V^{p,w-p}$ as the Hodge grading and we say the level of the variation is the difference $p_{\max} - p_{\min}$ where $p_{\max}$ (resp. $p_{\min}$) is the maximum (resp. minimum) Hodge degree p for which $V^{p,w-p} \neq 0$. Observe that the level of a tensor product $V \otimes W$ is the sum of the levels of V and W. A polarization of the variation is a flat hermitian form h for which the splitting is orthogonal and $(-1)^p h$ is positive definite on $V^{p,w-q}$. In this case $F'^{w-p} = (F^{p+1})^\perp$. A variation which admits a polarization is said to be polarizable. We define $\mathbb{C}(-r,-s)$ to be the polarizable complex Hodge structure on $V = \mathbb{C}$ of weight $r+s$ with $V^{r,s} = V$.

Recall that the category of polarizable complex variations of Hodge structures is semi-simple. For the remainder we assume the base is compactfiable. Then for two polarizable complex variations V, W, the theorem of the fixed part [23, (7.22) Theorem] (applied to $\mathcal{H}\mathrm{om}(V,W)$) says that the group $\mathrm{Hom}(V,W)$ of morphisms of local systems has a natural complex Hodge structure whose degree (r,s) part is exactly the morphisms of complex variations $V \to W(r,s)$. We have the following further consequence due to Deligne:

Theorem 3 ([8, 1.13 Proposition]). *Suppose V is a $\mathbb{C}$-local system underlying a polarizable complex variation of Hodge structures on a compactifiable base and that we have a splitting of $\mathbb{C}$-local systems*

$$V = \bigoplus_i N_i \otimes A_i \tag{1}$$

where the N_i are irreducible and pairwise non-isomorphic and the A_i are nonzero complex vector spaces. Then

1. *Each N_i underlies a polarizable complex variation of Hodge structures, unique up to tensoring by $\mathbb{C}(r,s)$ (that is, shifting the bigrading).*
2. *Each polarizable complex variation of Hodge structures with underlying local system V arises from (1) by equipping each N_i with its unique (up to shifts) polarizable complex variation of Hodge structures and each A_i with a uniquely determined polarizable complex Hodge structure, namely $A_i = \mathrm{Hom}(N_i, V)$.*

In particular, the theorem implies a polarizable complex variation is irreducible if and only if the underlying local system is, and so we may unambiguously speak of the irreducible factors of a polarizable complex variation.

Remark 4. The $\mathbb{C}$-local system V underlying a polarizable complex variation of Hodge structures on an algebraic space is semi-simple, see [8, §1.12]. This is a consequence of an unpublished result of Nori (see [8, §1.12]) and uses that the orthogonal complement of flat subbundle with respect to the Hodge metric (which is a harmonic metric) is flat. In our setting, we will only need to apply this to V underlying a polarizable integral variation of Hodge structures. In this case, the semi-simplicity of the underlying $\mathbb{Q}$-local system is a result of Deligne [9, 4.2.6]. Note that for a perfect field K and a field extension $K \subset L$, a K-local system V is semi-simple if and only if V_L is, as semi-simplicity is equivalent to being generated by simple sub-local systems, see for example [5, §12.7].

Given an $\mathbb{R}$-local system V, a polarizable real variation of Hodge structures of weight w on V in the usual sense naturally induces a polarizable complex variation of weight w on $V_{\mathbb{C}}$. Conversely, a polarizable complex variation of weight w on $V_{\mathbb{C}}$ comes from a polarizable real variation on V if complex conjugation flips the Hodge grading, or more precisely if for some (hence any) polarization h the isomorphism of local systems $V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee$ given by $y \mapsto h(-,\overline{y})$ induces an isomorphism of complex variations $V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee(-w)$. Indeed, if this is the case then $V^{p,w-p} \xrightarrow{\cong} (V^{w-p,p})^\vee$ so $\overline{V^{p,w-p}} = V^{w-p,p}$. Moreover, for even w (resp. odd w) a real polarization is provided by the symmetric (resp. antisymmetric) real form $q(x,y) = h(x,\overline{y})+h(y,\overline{x})$ (resp. $q(x,y) = i(h(x,\overline{y})-h(y,\overline{x}))$), since $q(x,\overline{x}) = h(x,x) + h(\overline{x},\overline{x})$ (resp. $-iq(x,\overline{x}) = h(x,x) - h(\overline{x},\overline{x})$) is definite of alternating sign on $V^{p,w-p}$.

The category of polarizable real variations is also semi-simple. Observe that by Theorem 3 any isotypic component W of an $\mathbb{R}$-local system V underlying a polarizable real variation is canonically a real subvariation, as the same is true over $\mathbb{C}$ and the isomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}^\vee(-w)$ coming from h restricts to an isomorphism $W_{\mathbb{C}} \to W_{\mathbb{C}}^\vee(-w)$. If V is a single isotypic factor, then $V_{\mathbb{C}}$ either has one self-conjugate irreducible factor N or has two non-isomorphic conjugate irreducible factors $N, \overline{N}$. Note that $N^\vee \cong \overline{N}$ as local systems via the polarization. Note also that the level of a polarizable real variation V is at least as large as the level of any of the irreducible factors of $V_{\mathbb{C}}$.

We say that a real or complex variation is *isotrivial* if the Hodge filtration is flat, or equivalently if the irreducible factors of the complexification are level zero[1]. To summarize the above discussion:

Lemma 5. *Let V be an irreducible polarizable real variation of Hodge structures of level one. Then V is either isotrivial, or every irreducible factor of $V_{\mathbb{C}}$ is level one.*

Before turning to the proof of Theorem 1 we recall the definition of a primitive symplectic variety. A complex analytic variety X is a symplectic variety in the sense of Beauville[2] [3] if it has rational singularities and a nowhere degenerate 2-form σ on its regular locus X^{reg}. A primitive symplectic variety X is a compact Kähler symplectic variety such that $H^1(X, \mathcal{O}_X) = 0$ and $H^0(X^{\mathrm{reg}}, \Omega^2_{X^{\mathrm{reg}}}) = \mathbb{C}\sigma$. As the singularities are rational, for any resolution $\pi : Y \to X$ the form σ extends to a two-form on Y [15, Corollary 1.7]. Moreover, $\pi^* : H^2(X,\mathbb{Q}) \to H^2(Y,\mathbb{Q})$ is injective, so the Hodge structure on $H^2(X,\mathbb{Q})$ is pure, and we have an induced isomorphism $\pi^* : H^{2,0}(X) \to H^{2,0}(Y)$ (see [2] for details). In particular we have a well defined class $[\sigma] \in H^{2,0}(X)$.

By a Lagrangian fibration of a primitive symplectic variety we still mean a proper morphism $f : X \to B$ to a normal compact analytic variety B whose generic fiber is

[1] Or equivalently, if the monodromy is unitary (by Theorem 3); since there may not be an integral structure, this does not necessarily mean the monodromy is finite.

[2] This definition is equivalent to Beauville's original one, by the results of [15].

smooth, connected, and Lagrangian. Each smooth fiber will still be an abelian variety, by Voisin's argument.

Lemma 6. *For $f : X \to B$ a Lagrangian fibration of a primitive symplectic variety, B is projective.*

Proof. First, X has no nonzero 1-forms and a unique 2-form up to scaling (which is not pulled back from B), so B cannot admit nonzero 1-forms or 2-forms on its regular locus. Moreover, B is Kähler by applying [25, Corollaire to Théorème 3] since f is equidimensional by Matsushita's argument [20, Theorem 1] (see also [14, Lemma 1.17]) now using functorial pullback of reflexive forms [15, Theorem 1.11] and the fact that $R\pi_*\omega_Y \cong \omega_X \cong \mathcal{O}_X$ by the rationality of the singularities of X [17, §5.1]. Putting these two things together, B is Moishezon (as it admits a projective resolution), so an algebraic space. By Saito's decomposition theorem and the rationality of the singularities of X, $R(f\circ\pi)_*\mathcal{O}_Y = R(f\circ\pi)_*\omega_Y$ is split, so by a theorem of Kovács [18, Theorem 1] this implies the singularities of B are rational. It then follows B admits a line bundle L whose cohomology class is a Kähler class, and by Nakai–Moishezon for algebraic spaces (see for example [16, Theorem 3.11]) L is ample, hence B is projective. □

We use the same notation as above: $B^\circ \subset B$ is a dense Zariski open smooth subset over which the restriction $f^\circ : X^\circ \to B^\circ$ is smooth and $\varphi : B^\circ \to S$ is the period map associated to the variation of (polarized) weight one integral Hodge structures on B° with underlying local system $R^1 f^\circ_* \mathbb{Z}_{X^\circ}$.

Proof of Theorem 1. Let $V_\mathbb{Z} := R^1 f^\circ_* \mathbb{Z}_{X^\circ}$. We start with the following result of Voisin, whose proof we give for convenience (and to extend it slightly).

Lemma 7 ([26, Lemma 5.5])**.** *$V_\mathbb{R}$ is irreducible as a polarizable real variation of Hodge structures.*

Proof. First assume X is smooth. By a result of Matsushita [19, Lemma 2.2] the restriction map $H^2(X,\mathbb{R}) \to H^2(X_b,\mathbb{R})$ to a generic fiber of f° is rank one and by Deligne's global invariant cycles theorem $H^2(X,\mathbb{R}) \to H^0(B^\circ, R^2 f^\circ_*\mathbb{R}_{X^\circ})$ is surjective [9]. If $V_\mathbb{R}$ splits as a variation then the polarizations of the factors would yield a larger than one-dimensional space of sections of $R^2 f^\circ_*\mathbb{R}_{X^\circ} = \wedge^2 V_\mathbb{R}$, which is a contradiction.

Now if X is a $\mathbb{Q}$-factorial terminal[3] primitive symplectic variety, one easily checks using the results of [2] that Matsushita's proof carries through verbatim and that $H^2(X,\mathbb{R}) \to H^0(B^\circ, R^2 f^\circ_*\mathbb{R}_{X^\circ})$ is still surjective, since the cokernel of $\pi^* : H^2(X,\mathbb{R}) \to H^2(Y,\mathbb{R})$ is generated by exceptional divisors for a log resolution $\pi : Y \to X$. □

Suppose now that f is not of maximal variation. Define the real transcendental lattice $T \subset H^2(X,\mathbb{R})$ to be the polarizable real Hodge substructure spanned by $[\sigma]$ and $[\overline{\sigma}]$. We next claim that the polarizable real variation of Hodge structures $V_\mathbb{R} \otimes T^\vee$ has a nontrivial subvariation of level at most one after a finite base-change; the argument below is that of [24] at its core, with some mild modifications.

By trivializing sufficiently high level torsion of $V_\mathbb{Z}$, we obtain a finite Galois étale cover $\nu : B^{\circ\prime} \to B^\circ$ for which the pull-back $V'_\mathbb{Z} := \nu^* V_\mathbb{Z}$ is pulled back along its period map $\varphi' : B^{\circ\prime} \to S'$, where $S' \to S$ is the corresponding finite cover of S. Note that up to replacing $B^{\circ\prime}$ with a further finite Galois étale cover (so as to make the local monodromy unipotent), we may assume φ' can be embedded in a proper

[3] In fact, we only use the $\mathbb{Q}$-factoriality, and this is the only time the singularity assumption is used. Moreover, since a Lagrangian fibration of a primitive symplectic variety induces one on its $\mathbb{Q}$-factorial terminalization, we may deduce Theorem 1 without the singularity assumption provided X admits a $\mathbb{Q}$-factorial terminalization, for instance if X is projective.

map $\overline{\varphi}' : \overline{B}^{\circ\prime} \to S'$ [12, Propositions 9.10 and 9.11]. Denote by $\overline{B}^{\circ\prime} \to Z \to S'$ the Stein factorization of $\overline{\varphi}'$, by $\psi : B^{\circ\prime} \to Z$ the resulting map, and by $V''_{\mathbb{Z}}$ the variation on Z so that $V'_{\mathbb{Z}} = \psi^* V''_{\mathbb{Z}}$. The map $\overline{\varphi}'$ and its image Z are in fact algebraic [4, Theorem 3.1]. We shrink Z (and $B^\circ, B^{\circ\prime}, X^\circ$) so that it is smooth and so that $R^1\psi_*\mathbb{R}_{B^{\circ\prime}}$ is a local system, naturally underlying a graded polarizable real variation of mixed Hodge structures by Saito's theory of mixed Hodge modules [21, 22]), since B is projective. Note that the only nonzero Hodge components of $R^1\psi_*\mathbb{R}_{B^{\circ\prime}}$ have bidegrees $(0,0),(1,0),(0,1),(1,1)$, as the same is true of the fibers and base-change holds on an open set. Let $f^{\circ\prime} : X^{\circ\prime} \to B^{\circ\prime}$ be the base-change of f.

Consider the natural restriction map $\eta : \mathrm{pt}_Z^* T \to R^2(\psi \circ f^{\circ\prime})_*\mathbb{R}_{X^{\circ\prime}}$ along with the Leray spectral sequence computing $R(\psi \circ f^{\circ\prime})_*\mathbb{R}_{X^{\circ\prime}} = R\psi_* R f^{\circ\prime}_*\mathbb{R}_{X^{\circ\prime}}$. The natural map $H^2(X,\mathbb{C}) \to H^0(B^{\circ\prime}, R^2 f^{\circ\prime}_*\mathbb{C}_{X^{\circ\prime}})$ sends $[\sigma]$ and hence $T_{\mathbb{C}}$ to zero since the fibers of f are Lagrangian. Thus, the Leray spectral sequence implies η factors through a nonzero morphism $\mathrm{pt}_Z^* T \to R^1\psi_* V'_{\mathbb{R}} \cong V''_{\mathbb{R}} \otimes R^1\psi_*\mathbb{R}_{B^{\circ\prime}}$ in the category of real variations of mixed Hodge structures. This map is nonzero, for otherwise the Leray spectral sequence would again imply that $[\sigma]$ is pulled back from Z, which is absurd since σ is symplectic and the fibers of $\psi \circ f^{\circ\prime}$ are greater than half-dimensional.

Thus, there is a nonzero morphism of real variations

$$\mathrm{gr}^W_{-1}\, \psi^*(R^1\psi_*\mathbb{R}_{B^{\circ\prime}})^\vee \to V'_{\mathbb{R}} \otimes T^\vee. \tag{2}$$

As the category of polarizable real variations of (pure) Hodge structures is semi-simple, we therefore have a splitting

$$V'_{\mathbb{R}} \otimes T^\vee = U \oplus W$$

of real variations, where $U \neq 0$ is the image of (2). In particular, U has level at most one and weight -1.

Now by Lemma 7 the Galois group of ν acts transitively on the isotypic factors of $V'_{\mathbb{R}}$. In particular, if f (and therefore $V_{\mathbb{R}}$) is not isotrivial, no factor of $V'_{\mathbb{R}}$ (as a variation) is isotrivial, or else its entire isotypic component would be, and so would $V_{\mathbb{R}}$. But then there can be no nonzero morphism of variations $V'_{\mathbb{R}} \otimes T^\vee \to U$. Indeed, by Lemma 5, an irreducible factor N of $V'_{\mathbb{C}}$ has level one of degrees $(1,0),(0,1)$, and $N \otimes T^\vee_{\mathbb{C}}$ can only map nontrivially to an irreducible factor of $U_{\mathbb{C}}$ of the form $N(1,1)$, while $\mathrm{Hom}(N \otimes T^\vee_{\mathbb{C}}, N(1,1)) = T_{\mathbb{C}}(1,1) \cong \mathbb{C}(-1,1) \oplus \mathbb{C}(1,-1)$ has no degree 0 elements. Thus, f must be isotrivial. □

Remark 8. Van Geemen and Voisin use the Künneth decompostion (which requires a projectivity assumption) in place of the Leray spectral sequence for the same step in their proof. We briefly describe this perspective as it is more geometric. Through a very general point $b \in B^{\circ\prime}$, say above a point $z \in Z$, let F be the positive-dimensional fiber of ψ through b. The restricted family $X_F \to F$ has trivial monodromy, and in the projective case we have $X_F \cong X_b \times F$ (possibly after a further base-change). We then have the diagram

$$\begin{tikzcd} T_{\mathbb{C}} \arrow[r] \arrow[d] & (V''_{\mathbb{C}} \otimes R^1\psi_*\mathbb{C}_{B^{\circ\prime}})_z \arrow[d] \\ H^2(X_F,\mathbb{C}) \arrow[r] & H^1(X_b,\mathbb{C}) \otimes H^1(F,\mathbb{C}) \end{tikzcd}$$

where the bottom map is the Künneth projection. The image of $[\sigma]$ is then nonzero in the bottom right corner as follows: (i) σ is nonzero when restricted to X_F since $\dim X_F > \frac{1}{2}\dim X$; (ii) $\sigma|_{X_F}$ extends to a smooth compactification since σ extends to a smooth compactification of X°, so $[\sigma] \neq 0 \in H^2(X_F,\mathbb{C})$; (iii) the image of $[\sigma]$ in

$H^2(X_b,\mathbb{C})\otimes H^0(F,\mathbb{C})$ vanishes and $[\sigma]$ is not in the image of $H^0(X_b,\mathbb{C})\otimes H^2(F,\mathbb{C})$, as it is not pulled back from F.

Example 9. We revisit the example [24, §4] from van Geemen and Voisin as well. Let $p\geq 5$ be a prime and λ a pth root of unity. Consider a family of abelian varieties $f: X\to B$ with a cyclic automorphism such that the induced automorphism α of $V_\mathbb{R}=R^1f_*\mathbb{R}_X$ has λ as an eigenvalue on $V^{1,0}$ but not on $V^{0,1}$. Let α' be the automorphism of $T^\vee$ with eigenvalue λ^{-1} on $(T^\vee)^{-2,0}$ and eigenvalue λ on $(T^\vee)^{0,-2}$. Then $V_\mathbb{R}\otimes T^\vee$ has a level one factor, namely the 1 eigenspace $(V_\mathbb{R}\otimes T^\vee)^1$ of $\alpha\otimes\alpha'$. But the condition on the eigenvalues means the eigenspaces $(V_\mathbb{C})^\lambda$ and $(V_\mathbb{C})^{\lambda^{-1}}$ are level zero, and the real variation $(V_\mathbb{C})^\lambda\oplus(V_\mathbb{C})^{\lambda^{-1}}$ is an isotrivial real factor.

We close by discussing an application of Theorem 1 using a result of Gao, which is a simple application of the Ax–Schanuel theorem for universal families of abelian varieties [11, Theorem 1.1]. Recall that for a projective family $f: X\to B$ of g-dimensional abelian varieties[4] equipped with a section s and letting $\tilde{B}\to B^{\mathrm{an}}$ be the universal cover, the Betti map $\beta:\tilde{B}\to H_1(X_b,\mathbb{R})$ is the real analytic map obtained by taking the coordinates of the section s with respect to the flat real-analytic trivialization of f. Observe that $\beta^{-1}(H_1(X_b,\mathbb{Q}))$ is the set of points of $\tilde{B}$ at which s is torsion. If $\varphi: B\to S$ is the period map of f and $\mathcal{X}\to S$ the universal family of abelian varieties, then s naturally yields a map $B\to\mathcal{X}$ lifting φ. We say that s is of maximal variation if $B\to\mathcal{X}$ is generically finite.

Proposition 10 ([11, Theorem 9.1])**.** *Let $f: X\to B$ be a projective family of g-dimensional abelian varieties with $\dim B\geq g$ and whose very general fiber has no nontrivial isogeny factor. Let $s: B\to X$ be a non-torsion section of f which is of maximal variation. Then the Betti map $\beta:\tilde{B}\to H_1(X_b,\mathbb{R})$ associated to s is generically submersive.*

Corollary 11. *Let X be a $\mathbb{Q}$-factorial terminal primitive symplectic variety and $f: X\to B$ a Lagrangian fibration. Let L be a line bundle whose restriction to the smooth fibers is topologically trivial. Then the set of points $b\in B^\circ(\mathbb{C})$ for which $L|_{X_b}$ is torsion is analytically dense in B.*

Corollary 11 was proven by Voisin [26, Theorem 1.3] assuming either f is of maximal variation and $\dim X\leq 8$ or isotrivial with no restriction on the dimension. Our use of Proposition 10 replaces the results of André–Corvaja–Zannier [1] in [26].

Proof of Corollary 11. By Voisin's result and Theorem 1 we may assume f is of maximal variation. Note that Voisin's proof works equally well in the singular case; we leave the details to the reader. Consider the family of abelian varieties $h:\mathrm{Pic}^0(X^\circ/B^\circ)\to B^\circ$ and the section $s: b\mapsto L|_{X_b}$. Let $\nu: B^{\circ\prime}\to B^\circ$ be a Galois finite base-change for which the base-change of h is isogenous to a product of families whose very general fibers are simple. As the Galois group of ν acts transitively on the isogeny factors by Lemma 7, the d isogeny factors all have the same dimension g', and the image of the period map of each factor must have dimension $\geq g'$, or else the image of the period map of f would have dimension smaller than $dg'=\dim(X^\circ/B^\circ)=\dim(B^\circ)$. The base-change of the section s is also Galois invariant, so it suffices to prove the density statement for its projection to a single simple factor $Y^\circ\to B^{\circ\prime}$. Applying Proposition 10, the Betti map $\beta:\tilde{B}\to H_1(Y_b,\mathbb{R})$ is submersive, so $\beta^{-1}(H_1(Y_b,\mathbb{Q}))$ is analytically dense in $\tilde{B}$ as claimed. □

[4]Meaning X is an abelian scheme over B, so there is in particular a 0-section of f.

Corollary 11 has an interesting interpretation in terms of the Beauville conjecture for irreducible hyperkähler manifolds X, see the discussion in [26, §1.2]. There it is also shown how corollaries 2 and 11 can be used to construct constant cycle curves on K3 surfaces.

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