LECTURES ON THE AX–SCHANUEL CONJECTURE

BENJAMIN BAKKER AND JACOB TSIMERMAN

Abstract. Functional transcendence results have in the last decade found a number of important applications to the algebraic and arithmetic geometry of varieties X admitting flat or hyperbolic uniformizations: Pila and Zannier's new proof of the Manin–Mumford conjecture, the proof of the André–Oort conjecture for A_g, and the generic Shafarevich conjecture for hypersurfaces of Lawrence–Venkatesh, to name a few. The key insight (originally stemming from work of Pila and Zannier) is the use of o-minimality to pass between the geometry of X and that of its uniformizing space.

The goal of these lectures is to give a tour through the main elements of the proof of the Ax–Schanuel conjecture for variations of Hodge structures intended for non-experts. We start by introducing the basic notions of o-minimal geometry with a view towards the two algebraization theorems of Pila–Wilkie and Peterzil–Starchenko. We then show how these results are combined with local volume bounds in the style of Hwang–To to prove the Ax-Schanuel conjecture.

These notes originated from the lecture series by the authors at the workshop *Shimura varieties and hyperbolicity of moduli spaces*, UQAM (Montreal), May 28–June 1, 2018. The authors are grateful to the organizers for the invitation and for the wonderful conference.

Contents

1. Introduction to transcendence

1.1. Preliminaries.

We begin with some very basic definitions. For details on transcendence theory we refer to [Lan02, Chap. 8].

Definition 1.1.1. Let L/K be a field extension.

(1) For a finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset L$, an *algebraic relation over K* satisfied by $\{\alpha_1, \ldots, \alpha_n\}$ is a polynomial $p \in K[z_1, \ldots, z_n]$ such that

$$p(\alpha_1, \ldots, \alpha_n) = 0.$$

(2) $\alpha \in L$ is said to be *algebraic over* K if $\{\alpha\}$ satisfies a nonzero algebraic relation over K.

We will often use "$\{\alpha_1, \ldots, \alpha_n\}$ satisfies an algebraic relation over K" interchangeably with "$\alpha_1, \ldots, \alpha_n$ satisfy an algebraic relation over K".

Lemma 1.1.2. *Let L/K be a field extension.*

(1) $\alpha \in L$ is algebraic over K if and only if there is a finite-dimensional K-vector subspace $V \subset L$ with $\alpha V \subset V$.
(2) If $\beta_1, \ldots, \beta_n \in L$ are algebraic over K and $\alpha \in L$ such that $\{\alpha, \beta_1, \ldots, \beta_n\}$ satisfies an algebraic relation over K that is nonconstant in α then α is algebraic over K.
(3) The set $F \subset L$ of elements which are algebraic over K is a subfield.

Definition 1.1.3. Let L/K be a field extension.

(1) A finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset L$ is *algebraically independent over* K if it satisfies no nonzero algebraic relation over K. A subset $\Sigma \subset L$ is *algebraically independent over* K if every finite subset is algebraically independent.
(2) $\alpha \in L$ is *transcendental over* K if $\{\alpha\}$ is algebraically independent over K.
(3) A *transcendence basis* for L over K is a maximal subset of L which is algebraically independent over K.

We will often use "$\{\alpha_1, \ldots, \alpha_n\}$ is algebraically independent over K" interchangeably with "$\alpha_1, \ldots, \alpha_n$ are algebraically independent over K".

Example 1.1.4. $e \in \mathbb{R}$ is transcendental over $\mathbb{Q}$, as is $\pi \in \mathbb{R}$.

Example 1.1.5. It is conjectured but not known that $\{e, \pi\} \subset \mathbb{R}$ is algebraically independent over $\mathbb{Q}$.

Lemma 1.1.6. *Any two transcendence bases of L/K have the same cardinality.*

Definition 1.1.7. The *transcendence degree of L over K*, denoted $\operatorname{trdeg}_K L$, is the cardinality of a transcendence basis of L over K.

Example 1.1.8. For any field K, it is easy to see that any nonconstant $f \in K(t)$ is transcendental over K and moreover that $\{f\}$ is a transcendence basis of $K(t)$ over K. Thus, $\operatorname{trdeg}_K K(t) = 1$.

Example 1.1.9. The transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is equal to the cardinality of $\mathbb{C}$.

1.2. Classical transcendence of the exponential function.

Arithmetic transcendence. Naively we think of the exponential function e^z as highly transcendental. By this we mean that given $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, we expect algebraic relations among the arguments α_i to rarely translate into algebraic relations among the values e^{α_i}, and vice versa. There is one notable exception: since the exponential function is a group homomorphism $\mathbb{C} \to \mathbb{C}^*$, where $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$, any $\mathbb{Q}$-linear relation

$$0 = r_1\alpha_1 + \cdots + r_n\alpha_n \quad \text{for} \quad r_i \in \mathbb{Q}$$

leads to a "trivial" algebraic relation

$$1 = (e^{\alpha_1})^{r_1 b} \cdots (e^{\alpha_n})^{r_n b}$$

where $r_i = a_i/b_i$ in lowest terms and $b = \operatorname{lcm}(b_1, \ldots, b_n)$.

If we assume $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ satisfy no $\mathbb{Q}$-linear relations, we have the following longstanding conjecture:

Conjecture 1.2.1 (Schanuel Conjecture)**.** *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be $\mathbb{Q}$-linearly independent. Then*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n \tag{1}$$

Note that the conjecture is only interesting when the α_i are algebraically dependent—it is a statement about how algebraic relations among the α_i interact with algebraic relations with the exponentials.

The Schanuel conjecture remains wide open; to give a sense of how strong it is, we have the following example.

Example 1.2.2. Take $\alpha_1 = 1$ and $\alpha_2 = \pi i$. Then the conjecture implies

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(1, \pi i, e, -1) = \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\pi, e) \geq 2$$

that is, that e and π are algebraically independent over $\mathbb{Q}$.

By taking $\alpha_i \in \overline{\mathbb{Q}}$, we see that the statement of Schanuel's conjecture is optimal, since

$$n \geq \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n}) = \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}).$$

Moreover, in this case the conjecture says that $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$, and this has in fact been verified:

Theorem 1.2.3 (Lindemann–Weierstrass)**.** *Let $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ be $\mathbb{Q}$-linearly independent. Then*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(e^{\alpha_1}, \ldots, e^{\alpha_n}) = n.$$

Formal functional transcendence. The exponential function is also defined on formal power series $f \in \mathbb{C}[[t_1, \ldots, t_m]]$, and we may try to obtain functional analogs of the above arithmetic statements by simply replacing the extension $\mathbb{C}/\mathbb{Q}$ by $\mathbb{C}((t_1, \ldots, t_m))/\mathbb{C}$. Given $f_1, \ldots, f_n \in \mathbb{C}[[t_1, \ldots, t_m]]$, both sides of (1) have a clear analog, and we might guess that the correct statement is

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n}) \geq n.$$

There are however several new phenomena we must take into account. First, any relation of the form

$$\zeta = r_1 f_1 + \cdots + r_n f_n \quad \text{for} \quad r_i \in \mathbb{Q} \text{ and } \zeta \in \mathbb{C}$$

leads to a "trivial" algebraic relation among the exponentials $e^{f_1}, \ldots, e^{f_n}$, so we should assume the f_i are $\mathbb{Q}$-linearly independent *modulo constant terms.*

Second, the f_i may now satisfy a *formal* relation $p \in \mathbb{C}[[z_1, \ldots, z_n]]$, meaning that

$$0 = p(f_1, \ldots, f_n).$$

Example 1.2.4. Not surprisingly, the f_i may be algebraically independent over $\mathbb{C}$ and still satisfy a formal relation. Indeed, $f_1 = t$ and $f_2 = e^t$ satisfy a formal relation, namely

$$p(z_1, z_2) = e^{z_1} - z_2.$$

Formal relations are in fact much easier to detect. By the formal implicit function theorem, the number of independent formal relations is encoded by the dimension of the kernel of the Jacobian matrix $J(f_1, \ldots, f_n) := \left(\frac{\partial f_i}{\partial t_j}\right)$ over $\mathbb{C}((t_1, \ldots, t_m))$, and the *formal* transcendence degree can be reasonably defined to be the rank of $J(f_1,\dots,f_n)$. The correct analog of Conjecture 1.2.1— which is a theorem due to Ax [Ax71]—says roughly that the *algebraic* transcendence degree of $\mathbb{C}(f_1,\dots,f_n,e^{f_1},\dots,e^{f_n})$ over $\mathbb{C}$ is at least n *more* than the *formal* transcendence degree of the f_i:

Theorem 1.2.5 (Ax–Schanuel, Theorem 3 of [Ax71])**.** *Let* $f_1,\dots,f_n\in\mathbb{C}[[t_1,\dots,t_m]]$ *be* $\mathbb{Q}$*-linearly independent modulo* $\mathbb{C}$*. Then*

$$\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(f_1,\dots,f_n,e^{f_1},\dots,e^{f_n})\geq n+\operatorname{rk}J(f_1,\dots,f_n). \tag{2}$$

Of course, we always have

$$\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(f_1,\dots,f_n)+\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(e^{f_1},\dots,e^{f_n})\geq\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(f_1,\dots,f_n,e^{f_1},\dots,e^{f_n})$$

from which we deduce the following weaker version, which is often what's used in applications.

Corollary 1.2.6 (Weak Ax–Schanuel)**.** *In the setup of Theorem 1.2.5, we have*

$$\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(f_1,\dots,f_n)+\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(e^{f_1},\dots,e^{f_n})\geq n+\operatorname{rk}J(f_1,\dots,f_n). \tag{3}$$

As a further corollary, we can deduce an analog of the Lindemann–Weierstrass theorem:

Corollary 1.2.7 (Ax–Lindemann–Weierstrass)**.** *In the setup of Theorem 1.2.5, further assume*

$$\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(f_1,\dots,f_n)=\operatorname{rk}J(f_1,\dots,f_n). \tag{4}$$

Then

$$\operatorname{trdeg}_{\mathbb{C}}\mathbb{C}(e^{f_1},\dots,e^{f_n})=n.$$

Condition (4) has a clear geometric interpretation: if the f_i converge in some ball centered at the origin, it means the image of the germ $(f_1,\dots,f_n):\mathbb{C}^m\to\mathbb{C}^n$ is (the germ of) an algebraic variety. This observation naturally leads us to the geometric approach of the next section.

Geometric functional transcendence. Often a more geometric interpretation of the results of the previous section admits clearer generalizations to other settings. The key point is that if we replace the field $\mathbb{C}((t_1,\dots,t_m))$ from the previous section with the subfield $\mathbb{C}\langle\langle t_1,\dots,t_m\rangle\rangle\subset\mathbb{C}((t_1,\dots,t_m))$ of power series that converge in some ball around the origin, it does not affect the transcendence statements (see [Sei58]).

Now, transcendence statements about the field of convergent power series can be phrased in terms of the analytic varieties they parametrize. For example, consider the flat uniformization

$$\pi:\mathbb{C}^n\to(\mathbb{C}^*)^n:(z_1,\dots,z_n)\mapsto(e(z_1),\dots,e(z_n))$$

where[1] $e(z)=e^{2\pi iz}$. Both $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ can be endowed with obvious structures as complex algebraic varieties, and it is then natural to ask what algebraic subvarieties $L\subset\mathbb{C}^n$ also have algebraic "image". To formulate this precisely, for a complex algebraic variety X and a subset $Y\subset X$, we denote by Y^{Zar} the Zariski closure of Y in X. We make the following definition:

Definition 1.2.8. We say an algebraic subvariety $L\subset\mathbb{C}^n$ is *bialgebraic* if

$$\dim L=\dim\pi(L)^{\mathrm{Zar}}.$$

In this case we will sometimes abusively refer to $\pi(L)$ as being bialgebraic as well.

[1]We could formulate everything with $e(z)=e^z$ and the statements would be identical.

Example 1.2.9. Building on the "trivial" algebraic relations from the previous subsection, any $L \subset \mathbb{C}^n$ which is a $\mathbb{C}$-translate of a linear subspace of $\mathbb{C}^n$ defined over $\mathbb{Q}$ is bialgebraic. Said differently, every coset $M \subset (\mathbb{C}^*)^n$ of an algebraic subgroup of $(\mathbb{C}^*)^n$ is bialgebraic.

In fact, cosets of subtori are the *only* bialgebraic subvarieties, as we shall show in Corollary 4.1.2:

Proposition 1.2.10 (See Corollary 4.1.2)**.** *Every closed bialgebraic $M \subset (\mathbb{C}^*)^n$ is a finite union of cosets of subtori.*

Now consider the following situation. Let $W \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$ be the graph of π, and let $\mathrm{pr}_1 : \mathbb{C}^n \times (\mathbb{C}^*)^n \to \mathbb{C}^n$ be the first projection. Suppose we have an algebraic subvariety $V \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$, as well as an analytic component U of the intersection $V \cap W$. Let $\Delta \subset \mathbb{C}$ be the unit disk. Taking a local holomorphic parametrization $f = (f_1, \ldots, f_n) : \Delta^m \to \mathrm{pr}_1(U) \subset \mathbb{C}^n$, we see that on the one hand

$$\mathrm{rk}\, J(f_1, \ldots, f_n) = \dim \mathrm{pr}_1(U) = \dim U$$

while on the other hand if we consider the formal power series expansions at the origin $f_i \in \mathbb{C}[[t_1, \ldots, t_m]]$,

$$\mathrm{trdeg}_{\mathbb{C}}\, \mathbb{C}(f_1, \ldots, f_n, e(f_1), \ldots, e(f_n)) = \dim U^{\mathrm{Zar}} \leq \dim V.$$

Given Example 1.2.9, for $f_1, \ldots, f_n$ to be $\mathbb{Q}$-linearly independent modulo constant terms, we equivalently must have that $\mathrm{pr}_1(U)$ is not contained in any proper bialgebraic subvariety $L \subset \mathbb{C}^n$, in which case Theorem 1.2.5 says that we must have

$$\dim V \geq n + \dim U. \tag{5}$$

Conversely, suppose that for any algebraic $V \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$ and any analytic component U of $V \cap W$ that is not contained in the graph of a proper bialgebraic subvariety we have (5). Then given a holomorphic function $f = (f_1, \ldots, f_n) : \Delta^m \to \mathbb{C}^n$ whose image is not contained in any bialgebraic subvariety, define

$$F = f \times (\pi \circ f) : \Delta^m \to \mathbb{C}^n \times (\mathbb{C}^*)^n$$

and take $V = F(\Delta^m)^{\mathrm{Zar}}$, so that

$$\mathrm{trdeg}_{\mathbb{C}}\, \mathbb{C}(f_1, \ldots, f_n, e(f_1), \ldots, e(f_n)) = \dim V.$$

Some analytic component U of the intersection $V \cap W$ must contain $F(\Delta^m)$, and U cannot be contained in the graph of a proper bialgebraic subvariety by the assumption on $f(\Delta^m)$, so (5) would imply

$$\dim V \geq n + \dim U \geq n + \dim F(\Delta^m) = n + \mathrm{rk}\, J(f_1, \ldots, f_n).$$

Rephrasing, we have therefore proven the following statement is equivalent to Theorem 1.2.5:

Theorem 1.2.11 (Ax–Schanuel)**.** *Let $W \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$ be the graph of π, and suppose there is an algebraic subvariety $V \subset \mathbb{C}^n \times (\mathbb{C}^*)^n$ such that there is an analytic component U of $V \cap W$ of unexpected codimension:*

$$\mathrm{codim}_{\mathbb{C}^n \times (\mathbb{C}^*)^n}(U) < \mathrm{codim}_{\mathbb{C}^n \times (\mathbb{C}^*)^n}(V) + \mathrm{codim}_{\mathbb{C}^n \times (\mathbb{C}^*)^n}(W).$$

Then U is contained in the graph of a proper bialgebraic $L \subset \mathbb{C}^n$.

The moral is that "atypical" intersections between algebraic subvarieties of $\mathbb{C}^n \times (\mathbb{C}^*)^{\mathrm{n}}$ and the graph of π are controlled by bialgebraic subvarieites.

We of course also have geometric versions of Corollaries 1.2.6 and 1.2.7:

Corollary 1.2.12 (Weak Ax–Schanuel)**.** *Suppose there are algebraic subvarieties $V_1 \subset \mathbb{C}^n$ and $V_2 \subset (\mathbb{C}^*)^n$ such that there is an analytic component U of $V_1 \cap \pi^{-1}(V_2)$ of unexpected codimension. Then U is contained in a proper bialgebraic $L \subset \mathbb{C}^n$.*

Proof. Take $V = V_1 \times V_2$. □

Corollary 1.2.13 (Ax–Lindemann–Weierstrass)**.** *Suppose there are algebraic subvarieties $V_1 \subset \mathbb{C}^n$ and $V_2 \subset (\mathbb{C}^*)^n$.*

(1) If $\pi(V_1) \subset V_2$, then there is a bialgebraic $M \subset (\mathbb{C}^*)^n$ with

$$\pi(V_1) \subset M \subset V_2;$$

(2) If $\pi(V_1) \supset V_2$, then there is a bialgebraic $M \subset (\mathbb{C}^*)^n$ with

$$\pi(V_1) \supset M \supset V_2.$$

Proof. For the first part, we have a containment $V_1 \subset \pi^{-1}(V_2)$ which is an intersection of unexpected codimension unless $V_2 = (\mathbb{C}^*)^n$. Thus, provided V_2 is a proper subvariety, by the previous corollary we obtain $L \subset \mathbb{C}^n$ bialgebraic containing V_1. Replacing $\mathbb{C}^n$ by L and V_2 by $\pi(L) \cap V_2$, we may continue until $\pi(L) \cap V_2 = \pi(L)$—that is, until $\pi(L) \subset V_2$.

We leave the second part as an exercise. □

Corollary 1.2.13 can be equivalently formulated as the following:

Corollary 1.2.14.

(1) For $V \subset \mathbb{C}^n$ algebraic, $\pi(V)^{\mathrm{Zar}} \subset (\mathbb{C}^*)^n$ is a finite union of cosets of subtori.

(2) For $V \subset (\mathbb{C}^*)^n$ algebraic and any component V_0 of $\pi^{-1}(V)$, we have that $V_0^{\mathrm{Zar}} \subset \mathbb{C}^n$ is a finite union of $\mathbb{C}$-translates of linear subspaces defined over $\mathbb{Q}$.

Note that it is really the first part of Corollary 1.2.14 that is the analog of Corollary 1.2.7. It can also be stated as:

Corollary 1.2.15. *For any closed algebraic $V \subset (\mathbb{C}^*)^n$, a maximal irreducible algebraic subvariety of $\pi^{-1}(V)$ is a coset of a subtorus.*

We leave it to the reader to show that Corollary 1.2.6 (resp. 1.2.7) is equivalent to Corollary 1.2.12 (resp. 1.2.13).

1.2.16. *Semiabelian varieties.* Let $Y = A$ be a semi-abelian variety with identity $0 \in Y$. Let $X = V$ be its universal cover with its natural structure as a complex vector space, $\pi : V \to A$ the covering map, and $\Lambda = \pi^{-1}(0)$, which is a discrete subgroup of V. The universal covering map π is then identified with the quotient map $V \to V/\Lambda$. Note that if we had started with V and $\Lambda \subset V$ a discrete subgroup, V/Λ is not guaranteed to have the structure of an algebraic variety, and if it does it may not be unique.

The bialgebraic $M \subset A$ are then cosets of algebraic subgroups of A, and the Ax–Schanuel conjecture was proven by Ax [Ax72].

In fact, more generally still, it makes sense to allow X, Y to be (euclidean) open subsets of algebraic subvarieties $\check{X}, \check{Y}$, in which case we proceed as above defining the "algebraic subvarieties" of X to be intersections $V \cap X$ for V an algebraic subvariety of $\check{X}$, and likewise for Y.

1.2.17. *Shimura varieties.* A Shimura variety is a quotient of a bounded symmetric domain by an arithmetic lattice in a semisimple algebraic group $\mathbf{G}$. We will discuss this case more precisely in Lecture 4; for now we just give an example:

Example 1.2.18. The (coarse) moduli space of principally polarized abelian varieties A_g is a Shimura variety. In this case A_g admits a uniformization $\pi : \mathbb{H}_g \to A_g$ realizing A_g as the quotient of Siegel space

$$\mathbb{H}_g := \{Z \in \mathrm{Mat}_{g\times g}(\mathbb{C}) \mid Z^t = Z \text{ and } \operatorname{Im} Z > 0\}$$

by the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ via

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

$\mathbb{H}_g$ is naturally a semialgebraic subset of its compact dual $\check{\mathbb{H}}_g$, which is the projective variety parametrizing Lagrangian planes in $\mathbb{C}^{2g}$.

The classification of bialgebraic subvarieties in Shimura varieties is known by [UY11]. These are the so-called weakly special subvarieties. The Ax–Lindemann–Weierstrass conjecture was proven by Pila for powers of the modular curve [Pil11], by Pila–Tsimerman for A_g [PT14], and then by Klingler–Ulmo–Yafaev for general Shimura varieties [KUY16]. The Ax–Schanuel conjecture was proven by Pila–Tsimerman [PT16] for powers of the modular curve and by Mok–Pila–Tsimerman for general Shimura varieties [MPT17].

Importantly, Shimura varieties are moduli spaces of polarized pure integral Hodge structures which admit an algebraic structure, see Lecture 5.

1.2.19. *Mixed Shimura varieties.* We will give fewer details in this case, but mixed Shimura varieties arise by allowing $\mathbf{G}$ to have a nontrivial unipotent radical. Mixed Shimura varieties are moduli spaces of graded polarized mixed integral Hodge structures which admit an algebraic structure.

Example 1.2.20. The (coarse) universal family of principally polarized abelian varieties X_g over A_g is a mixed Shimura variety. In this case X_g admits a uniformization $\pi : \mathbb{H}_g \times \mathbb{C}^g \to X_g$ realizing X_g as the quotient by a group Γ which is an extension of $\mathrm{Sp}_{2g}(\mathbb{Z})$ by $\mathbb{Z}^{2g}$.

The classification of bialgebraic subvarieties in mixed Shimura varieties is known by [Gao17], and both the Ax–Lindemann–Weierstrass conjecture for mixed Shimura varieties and the Ax-Schanuel conjecture for the universal abelian variety have been proven by Gao [Gao17, Gao18].

1.2.21. *Period spaces.* Generalizing the case of Shimura varieties in a different direction, period spaces $\Gamma\backslash D$ parametrize pure polarized integral Hodge structures. Importantly, in this case $\Gamma\backslash D$ does not in general admit an algebraic structure, so the setup must be slightly modified (see Lecture 6). The proof of the Ax–Schanuel theorem (see Theorem 6.1.1 below) will be the main focus of these notes.

1.3. **Arithmetic applications.**

1.3.1. *Special point problems.* Suppose given a uniformization $\pi : X \to Y$ as in the previous section. Often there is a “special” set of points $Y_{\mathrm{sp}} \subset Y$ which have an interesting arithmetic interpretation in Y and whose *preimages* in X also have a simple arithmetic description.

Example 1.3.2. As in Example 1.2.16, take $Y = V/\Lambda$ a semiabelian variety, $X = V$, and $\pi : X \to Y$ the quotient map. Then we take Y_{sp} to be the set of torsion points, and $\pi^{-1}(Y_{\mathrm{sp}}) = \Lambda_{\mathbb{Q}}$.

Example 1.3.3. As in Example 1.2.17, take $Y = A_g$ the coarse moduli space of principally polarized abelian varieties, $X = \mathbb{H}_g$ the Siegel upper halfplane, and $\pi : \Omega \to Y$ the quotient. We take Y_{sp} to be the set of points corresponding to abelian varieties with CM. In this case, $\pi^{-1}(Y_{\mathrm{sp}})$ are points of $\check{\mathbb{H}}_g$ valued in number fields of bounded degree, with certain Galois properties.

Question 1.3.4. For an algebraic subvariety $V \subset Y$, denote $V_{\mathrm{sp}} := V \cap Y_{\mathrm{sp}}$. For what V do we have

$$(V_{\mathrm{sp}})^{\mathrm{Zar}} = V?$$

In the above contexts we expect that answer to be: only when V is bialgebraic. The property in Question 1.3.4 is in fact usually more restrictive, only holding for what are called *special* subvarieties, while bialgebraic subvarieties often turn out to be *weakly special*. For example, For $Y = (\mathbb{C}^*)^n$ and Y_{sp} the torsion points, the irreducible weakly special subvarieties are cosets of subtori whereas the irreducible special subvarieties are *torsion* cosets of subtori.

Example 1.3.5. In the case of the exponential $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ with torsion points as the special points the above expectation is known as Lang's conjecture. Precisely: if $V \subset (\mathbb{C}^*)^n$ is an algebraic variety and V_{tor} is the set of torsion points on V, then Lang conjectures $(V_{\mathrm{tor}})^{\mathrm{Zar}}$ is a finite union of torsion cosets of subtori. For $n = 2$ this was proven by Lang [Lan02].

Example 1.3.6. For $\pi : \mathbb{C}^n \to Y$ the uniformization of an abelian variety with torsion points as the special points, this is known as the Manin–Mumford conjecture. Precisely: if $V \subset Y$ is an algebraic variety and V_{tor} is the set of torsion points on V, they conjectured that $(V_{\mathrm{tor}})^{\mathrm{Zar}}$ is a finite union of torsion cosets of abelian subvarieties. Both the general form of Lang's conjecture and the Manin–Mumford conjecture were proven by Raynaud [Ray83a, Ray83b].

Example 1.3.7. For $\pi : \Omega \to Y$ the uniformization of a Shimura variety, this is known as the André-Oort conjecture. Precisely, if $V \subset Y$ is an algebraic variety and V_{sp} is the set of special points on V, then they conjectured that $(V_{\mathrm{sp}})^{\mathrm{Zar}}$ is a finite union of special subvarieties. The conjecture was conditionally[2] proven in [KY14] and unconditionally for $Y = A_g$ by [Tsi18].

The proof of Raynaud proceeds by singling out a prime p and using different ingredients to deal with the "prime-to-p-parts" and "p-parts". For the former, Raynaud crucially uses the Frobenius at p in the Galois group. He observes that the Frobenius operator on prime-to-p roots of unity is closely related to the multiplication by p map (they are identical in the multiplicative case). This allows him to reduce from a variety X to $X \cap (p \cdot X)$, and conclude by induction. This argument is heavily relied upon in the conditional proof of André-Oort assuming the generalized Riemann hypothesis in [KY14]. For the "p-part" Raynaud proceeds using a p-adic deformation theory argument, which is generalized to the Shimura case by Moonen [Moo98], allowing him to establish certain cases of André-Oort unconditionally.

The general hyperbolic case requires new ideas, and the proof of Tsimerman [Tsi18] builds on a strategy developed by Pila-Zannier which critically uses the Ax–Lindemann–Weierstrass theorem [Pil11].

[2]Conditional on the generalized Riemann hypothesis.

The Zilber–Pink conjecture. There is a wider set of conjectures, due to Bombier–Masser–Zannier in the multiplicative setting and and Zilber–Pink more generally. Instead of only considering special points, one considers points of various 'degrees' of specialness, and studies algebraic relations between such points. It is easiest to present in the multiplicative setting: for a point $x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n$ define its rank $\mathrm{rk}(x)$ to be the rank as an abelian group of the span $\langle x_1, \ldots, x_n \rangle$ in $\mathbb{C}^*$. Observe that the rank is 0 precisely for torsion points. One consequence of the conjecture is the following:

Conjecture 1.3.8 (Consequence of Zilber–Pink [Zil02, Pin05])**.** *Let $V \subset (\mathbb{C}^*)^n$ be an irreducible algebraic subvariety of codimension d. Let V_m be the points of $V(\mathbb{C})$ of rank at most m and assume V_{d-1} is Zariski-dense in V. Then V is contained in a proper special subvariety. In other words, there is a nonconstant monomial which is identically* 1 *on V.*

There is some progress on the conjecture above in the multiplicative case due to Habegger [Hab09], Maurin [Mau08], Bombieri–Masser–Zannier [BMZ08, BMZ99], and others. We refer the interested reader to [Pil14a] for a more complete survey.

The Shafarevich conjecture after Lawrence–Venkatesh. Lawrence and Venkatesh [LV18] have outlined a strategy for proving instances of the Shafarevich conjecture which uses the functional transcendence of period maps. Briefly, let $\mathcal{O} = \mathcal{O}_{K,S}$ be the ring of integers $\mathcal{O}_K$ in a number field K away from a finite set S of primes and $\pi : Y \to X$ a smooth projective family defined over $\mathcal{O}$. Then assuming certain geometric properties of π one expects the number of integral points $X(\mathcal{O})$ to be finite, for example when the family π has an immersive period map. The Shafarevich conjecture for moduli spaces of polarized abelian varieties was proven by Faltings in the landmark paper [Fal84].

The strategy of Lawrence–Venkatesh uses the p-adic period map in the context of p-adic Hodge theory. Their argument requires a p-adic transcendence result which formally follows from the corresponding transcendence result for the complex analytic period map. Using this technique, they are able to show that when X is taken to be certain moduli spaces of hypersurfaces in $\mathbb{P}^n$, the integral points $X(\mathcal{O})$ are not Zariski dense in X.

2. o-minimal geometry

For background on o-minimal structures and o-minimal geometry, we refer to [vdD98].

2.1. **o-minimal structures.**

An o-minimal structure specifies "tame" subsets of euclidean space which can be used as local models for "tame" geometry. On the one hand, the tameness will rule out pathologies such as Cantor sets and space-filling curves; on the other hand, as we will see, the tameness hypothesis locally imposes remarkably few conditions on analytic functions.

Definition 2.1.1. A structure S is a collection $(S_n)_{n\in\mathrm{N}}$ where each S_n is a set of subsets of $\mathbb{R}^n$ satisfying the following conditions:

(1) Each S_n is closed under finite intersections, unions, and complements;
(2) The collection (S_n) is closed under finite Cartesian products and coordinate projection;

(3) For every polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$, the zero set

$$(P = 0) := \{x \in \mathbb{R}^n \mid P(x) = 0\} \subset \mathbb{R}^n$$

is an element[3] of S_n.

We refer to the elements $U \in S_n$ as S-definable subsets of $\mathbb{R}^n$. For $U \in S_n$, and $V \in S_m$, we say a map $f : U \to V$ of S-definable sets is S-definable if the graph is. When the structure S is clear from context, we will often just refer to "definable" sets and functions.

The definable sets should be thought of as the sets that are "constructible" within the theory. From the axioms, it is easy to prove the following:

Proposition 2.1.2. *Let S be a structure.*

(1) The image and preimage of a definable set under a definable map is definable;
(2) The composition of two definable maps is definable.

Thus, for example, whereas we only required coordinate projections to be definable in Definition 2.1.1, it follows that all linear projections are definable. By definition, any structure S contains all real algebraic sets, but this is not enough:

Example 2.1.3. The collection S of real algebraic sets—that is, S_n = the Boolean algebra generated by sets of the form $(P = 0)$ for $P \in \mathbb{R}[x_1, \ldots, x_n]$—is *not* a structure. Indeed, for any $P \in \mathbb{R}[x_1, \ldots, x_n]$, the image of the projection of $(x_0^2 = P)$ forgetting x_0 is $(P \geq 0)$.

Example 2.1.4. Let $\mathbb{R}_{\text{alg}}$ be the collection of real semi-algebraic subsets of $\mathbb{R}^n$—that is, $(\mathbb{R}_{\text{alg}})_n$ is the Boolean algebra generated by sets of the form $(P \geq 0)$ for $P \in \mathbb{R}[x_1, \ldots, x_n]$. Then $\mathbb{R}_{\text{alg}}$ *is* a structure. By the Tarski–Seidenberg theorem (see for example [vdD98, Chapter 2]), coordinate projections of real semi-algebraic sets are real semi-algebraic, and the other axioms are easy to verify. $\mathbb{R}_{\text{alg}}$ is therefore a structure, in fact the structure generated by real algebraic sets given Example 2.1.3.

Remark 2.1.5. Tarski–Seidenberg is usually phrased as quantifier elimination for the real ordered field, and structures as defined above are important in model theory. Indeed, the axioms say definable sets are closed under first order formulas, as intersections, unions, and complements correspond to the logical operators "and", "or", and "not", while the projection axiom corresponds to universal and existential quantifiers. Moreover, we can make the same definition for any real closed field, and base-change to these fields plays a similar role to base-changing to generic points of schemes in algebraic geometry. We won't say much about it, but it is a useful perspective to keep in mind.

While infinite unions or intersections of definable subsets are not definable, it is nonetheless the case that many topological constructions with respect to the euclidean topology are definable:

Proposition 2.1.6. *Let S be a structure, and endow $\mathbb{R}^n$ with the euclidean topology. Closures, interiors, and boundaries of definable sets are definable.*

[3]One can work in greater generality by allowing structures without this assumption, but we will only require ones satisfying it.

Proof. We just show that the closure of a definable set $U \subset \mathbb{R}^n$ is defined by a first order formula and leave the rest as an exercise:

$$\overline{U} = \left\{ x \in \mathbb{R}^n \;\middle|\; \forall \epsilon > 0, \exists y \in U \text{ s.t. } \sum_i (x_i - y_i)^2 < \epsilon \right\}$$

□

Remark 2.1.7. We have the following formal operations on structures.

(1) Given two structures S and S', we say S is contained in S', denoted $S \subset S'$, if $S_n \subset S'_n$ for all n. Note that any structure S contains $\mathbb{R}_{\mathrm{alg}}$.
(2) Given structures $\{S^{(i)}\}_{i \in I}$ indexed by a set I, the intersection $(\bigcap S^{(i)})_n := \bigcap (S^{(i)})_n$ is evidently a structure. Thus, given a collection $(T_n)_{n \in \mathbb{N}}$ of sets of subsets of $\mathbb{R}^n$, we may speak of the structure S generated by the $(T_n)_{n \in \mathbb{N}}$ as the smallest structure S with $S_n \supset T_n$.
(3) Given an increasing chain
$$S^{(0)} \subset S^{(1)} \subset \cdots \subset S^{(i)} \subset \cdots$$
the union $(\bigcup S^{(i)})_n := \bigcup (S^{(i)})_n$ is a structure.

Thus far we have only specified the rules by which we can construct definable subsets from other definable subsets; we have not yet controlled how complicated definable sets are allowed to be. The crucial "tameness" property is o-minimality:

Definition 2.1.8. A structure S is said to be o-minimal if $S_1 = (\mathbb{R}_{\mathrm{alg}})_1$—that is, if the S-definable subsets of the real line are exactly finite unions of intervals.

The intervals in the definition are allowed to be closed or open on either end, may extend to infinity, and may be zero length (i.e. points).

Example 2.1.9. $\mathbb{R}_{\mathrm{alg}}$ is o-minimal, clearly.

Example 2.1.10. Let $\mathbb{R}_{\sin}$ be the structure generated by the graph of $\sin : \mathbb{R} \to \mathbb{R}$. $\mathbb{R}_{\sin}$ is not o-minimal as $\pi\mathbb{Z} = \sin^{-1}(0)$ is definable and infinite.

Example 2.1.11. Let $\mathbb{R}_{\exp}$ be the structure generated by the graph of the real exponential $\exp : \mathbb{R} \to \mathbb{R}$. $\mathbb{R}_{\exp}$ is o-minimal by a result of Wilkie [Wil99]. Quantifier elimination does not hold for $\mathbb{R}_{\exp}$.

Example 2.1.12. Let $\mathbb{R}_{\mathrm{an}}$ be the structure generated by the graphs of all restrictions $f|_{B(R)}$ of real analytic functions $f : B(R') \to \mathbb{R}$ on a finite radius $R' < \infty$ open euclidean ball (centered at the origin) to a strictly smaller radius $R < R'$ ball. Via the embedding $\mathbb{R}^n \subset \mathbb{RP}^n$, this is equivalent to the structure of subsets of $\mathbb{R}^n$ that are subanalytic in $\mathbb{RP}^n$. $\mathbb{R}_{\mathrm{an}}$ is o-minimal by van-den-Dries [vdD98], using Gabrielov's theorem of the complement. Note that while $\sin(x)$ is not $\mathbb{R}_{\mathrm{an}}$-definable, its restriction to any finite interval is.

Example 2.1.13. Let $\mathbb{R}_{\mathrm{an,exp}}$ be the structure generated by $\mathbb{R}_{\mathrm{an}}$ and $\mathbb{R}_{\exp}$. $\mathbb{R}_{\mathrm{an,exp}}$ is o-minimal by a result of van-den-Dries–Miller [vdDM96]. Most of the applications to algebraic geometry currently use the structure $\mathbb{R}_{\mathrm{an,exp}}$.

Remark 2.1.14. By Remark 2.1.7, there are maximal o-minimal structures, but not a unique one, as the structure generated by two o-minimal structures can fail to be o-minimal [RSW03].

For the rest of this lecture, we fix an o-minimal structure S, and by "definable" we mean S-definable, unless explicitly otherwise stated.

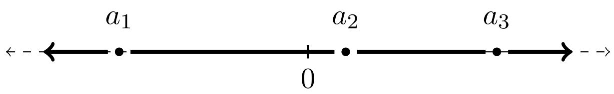


FIGURE 1. A cell decomposition of $\mathbb{R}$.

2.2. **Cylindrical cell decomposition.**

Sets that are definable in an o-minimal structure can be decomposed into graphs of definable functions in a systematic way. It would take us too far afield to prove the main existence result (Theorem 2.2.5 below), but it is important to keep in mind as it gives a clear picture of some of the finiteness properties that such definable sets possess.

We follow the treatment in [vdD98] closely.

Definition 2.2.1. A *definable cylindrical cell decomposition* of $\mathbb{R}^n$ is a partition $\mathbb{R}^n = \bigsqcup D_i$ into finitely many pairwise disjoint definable subsets D_i, called cells. The cells have the following inductive description.

$\underline{n = 0}$. There is exactly one definable cylindrical cell decomposition of $\mathbb{R}^0$. Its unique cell is all of $\mathbb{R}^0$.

$\underline{n > 0}$. Write $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. There is a definable cylindrical cell decomposition $\{E\}$ of $\mathbb{R}^{n-1}$ and for each E we have: an integer $m_E \in \mathbb{N}$ and continuous definable functions $f_{E,k} : E \to \mathbb{R}$ for each $0 < k < m_E$ such that

$$f_{E,0} := -\infty < f_{E,1} < \cdots < f_{E,m_E-1} < f_{E,m_E} := +\infty$$

The cells are:

- graphs: $\{(x, f_{E,k}(x)) \mid x \in E\}$ for each E and $0 < k < m_E$;
- bands: $(f_{E,k}, f_{E,k+1}) := \{(x, y) \mid x \in E \text{ and } y \in (f_{E,k}(x), f_{E,k+1}(x))\}$ for each E and $0 \leq k < m_E$.

Note that because of the inductive nature of the definition, we have implicitly chosen an ordering of the coordinates.

Example 2.2.2. The cylindrical cell decompositions of $\mathbb{R}$ are easy to understand. In this case, there is $m \in \mathbb{N}$ and $a_k \in \mathbb{R}$ for each $0 < k < m$ such that

$$a_0 := -\infty < a_1 < \cdots a_{m-1} < a_m := +\infty$$

and the cells are:

- $\{a_k\}$ for $0 < k < m$;
- (a_k, a_{k+1}) for $0 \leq k < m$.

Such a cell decomposition is shown in Figure 1.

Example 2.2.3. Figure 2.2 shows a cylindrical cell decomposition of $\mathbb{R}^2$ that projects to the cell decomposition of Figure 1.

Remark 2.2.4. Each cell D in a definable cylindrical cell decomposition has a well defined dimension $\dim_{\mathbb{R}} D$, and it is definably homeomorphic to $\mathbb{R}^{\dim_{\mathbb{R}} D}$ as follows. For $n = 0$ it is trivial, as it is inductively for the graph cells for $n > 0$. For band cells, given two definable $f, g : E \to \mathbb{R}$ with $f < g$, we have a definable homeomorphism $(f, g) \to E \times \mathbb{R}$ via

$$(x, y) \mapsto \left(x, \frac{1}{f(x) - y} + y + \frac{1}{g(x) - y}\right).$$

The main result is the following:

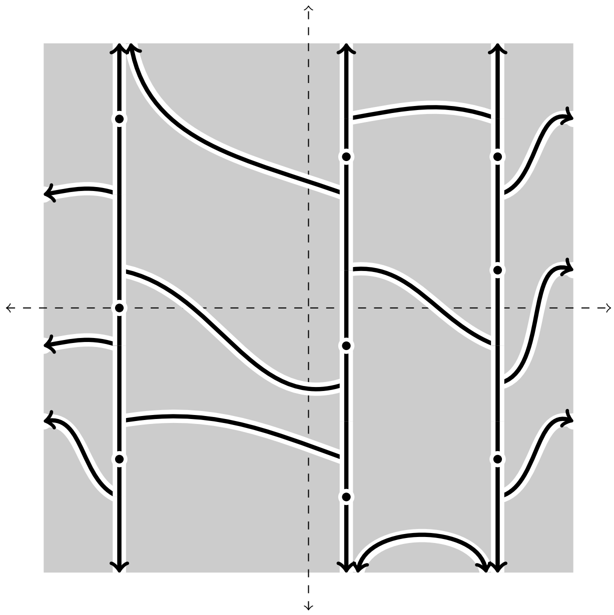


FIGURE 2. A cell decomposition of $\mathbb{R}^2$ projecting to that of Figure 1.

Theorem 2.2.5. *For any finite collection $U_j \subset \mathbb{R}^n$ of definable sets, there is a definable cylindrical cell decomposition of $\mathbb{R}^n$ such that each U_j is a union of cells.*

Every cell has a well-defined (real) dimension, so we have as a consequence:

Corollary/Definition 2.2.6. For any definable set $U \subset \mathbb{R}^n$ we define $\dim_{\mathbb{R}} U$ to be the largest dimension of its cells with respect to a definable cylindrical cell decomposition.

We won't give a proof of Theorem 2.2.5, but an essential ingredient is the following stronger version in a special case:

Lemma 2.2.7. For every definable function $f : (a, b) \to \mathbb{R}$, there is a finite subdivision

$$a_0 = a < a_1 < \cdots < a_m = b$$

such that each $f|_{(a_k,a_{k+1})}$ is either constant or strictly monotonic.

Proof. The proof is taken directly from [vdD98]. We begin with the following:

Claim. There is a subinterval $J \subset (a, b)$ on which f is constant or f is strictly monotonic and continuous.

Proof. We may assume f is not constant on any subinterval of (a, b).

Step 1. f is injective on a subinterval J.

It follows from the above assumption that all fibers are finite. The function $g(y) = \min f^{-1}(y)$ is a definable section of f, for we may write its graph as

$$\{(f(x), x) \in \mathbb{R}^2 \mid x \in (a, b) \text{ s.t. } x \leq x' \text{ for all } x' \in (a, b) \text{ with } f(x') = f(x)\}.$$

The image of g is definable and not finite by assumption, so by the o-minimality property it contains an interval J, and on this interval $g \circ f = \text{id}$, so f is injective.

Step 2. f is strictly monotonic on a subinterval J.

Assuming now that f is injective, for each $x \in (a,b)$ the sets

$$\{y \in (a,b) \mid f(y) < f(x)\}$$
$$\{y \in (a,b) \mid f(y) > f(x)\}$$

are a definable partition of $(a,b)\backslash\{x\}$. It follows that the sets

$$\begin{aligned} A =&\{x \in (a,b) \mid \exists\epsilon > 0 \text{ s.t. } f|_{(x-\epsilon,x)} < f(x) < f|_{(x,x+\epsilon)}\} \\ B =&\{x \in (a,b) \mid \exists\epsilon > 0 \text{ s.t. } f|_{(x-\epsilon,x)} > f(x) > f|_{(x,x+\epsilon)}\} \\ C =&\{x \in (a,b) \mid \exists\epsilon > 0 \text{ s.t. } f|_{(x-\epsilon,x)} > f(x) < f|_{(x,x+\epsilon)}\} \\ D =&\{x \in (a,b) \mid \exists\epsilon > 0 \text{ s.t. } f|_{(x-\epsilon,x)} < f(x) > f|_{(x,x+\epsilon)}\} \end{aligned}$$

are a definable partition of (a,b).

We now claim that the last two sets are finite; it's enough to show D is, as the proof for C is similar. If the claim were false, then there would be a subinterval J for which every point is a local maximum. For $n \in \mathbb{N}$, consider the sets

$$J_n := \{x \in J \mid x \text{ is a maximum on } (x - 1/n, x + 1/n)\}$$

which are clearly definable and $J = \cup_n J_n$. The J_n can't all be finite, so one J_n contains an interval by o-minimality, and this is clearly nonsense.

Thus, one of A and B (say A) contains an interval $J = (c,d)$. But then for each $x \in J$,

$$\{y \in J \mid y > x \text{ and } f|_{(x,y)} > f(x)\}$$

must be all of (x,d).

Step 3. f is strictly monotonic and continuous on a subinterval J.

Restrict f to an interval whose image is an interval. Then it is strictly monotonic and bijective, hence continuous. □

To finish, the set of points x for which either f is constant on a neighborhood of x or f is strictly monotonic and continuous in a neighborhood of x is definable, and hence is a finite set of points by the claim. This finishes the proof, since if for all x in some interval either f is constant on a neighborhood of x or f is strictly monotonic and continuous in a neighborhood of x, then the same is true on the entire interval. □

By reasoning along the lines of Lemma 2.2.7 one can show that definable functions have limits away from definable sets of smaller dimension. This can be upgraded to the fact that definable functions are C^k off of a definable set of smaller dimension:

Corollary 2.2.8. *Let $U \subset \mathbb{R}^n$ be a definable set. Then for each k, U has a stratification by definable C^k-submanifolds.*

Corollary 2.2.9. *Let $f : U \to V$ be a definable map. Then for each $n \in \mathbb{N}$, the subset*

$$V_n := \{v \in V \mid \dim f^{-1}(v) = n\} \subset V$$

is definable.

Proof. Consider the graph, and order the coordinates *backwards*. As is clear from the inductive definition, each cell has constant dimension over its projection. □

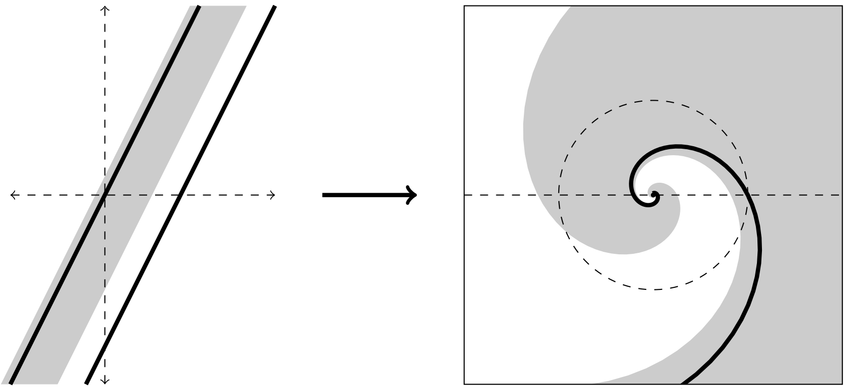


FIGURE 3. The "slanted strip" definable structures considered in Examples 2.3.2 and 2.3.3.

Corollary 2.2.10. *Let $f : U \to V$ be a definable map with finite fibers. Then for each $n \in \mathbb{N}$, the subset*

$$V_n := \{v \in V \mid \#f^{-1}(v) = n\} \subset V$$

is definable. Moreover, the size of the fibers is uniformly bounded.

Proof. As above, consider the graph and order the coordinates backwards. All of the cells are graphs over cells of V. □

2.3. Definable topological spaces.

Let M be a topological space. We can endow M with a geometry locally modeled on definable sets in the usual way using atlases.

Definition 2.3.1. A (S-)definable topological space M is a topological space M, a finite open covering V_i of M, and homeomorphisms $\varphi_i : V_i \to U_i \subset \mathbb{R}^n$ such that

(1) The U_i and the pairwise intersections $U_{ij} := \varphi_i(V_i \cap V_j)$ are definable sets;
(2) The transition functions $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : U_{ij} \to U_{ji}$ are definable.

We call the data (V_i, φ_i) a definable atlas. A morphism of definable spaces $f : M \to M'$ is a continuous map f such that for all i and i', the composition

$$(f \circ \varphi_i^{-1})^{-1}(V'_{i'}) \xrightarrow{\varphi_i^{-1}} f^{-1}(V'_{i'}) \xrightarrow{f} V'_{i'} \xrightarrow{\varphi'_{i'}} U'_{i'}$$

is S-definable. Note that this is a condition both on the map and the source. M is said to be a (S-)definable manifold if the definable atlas additionally gives M the structure of a manifold.

We denote the category of S-definable topoplogical spaces by (S-Top).

We will often use the term "(S-)definable structure" as a shorthand for "structure as a (S-)definable topological space" when no confusion is likely to arise, and likewise we will say a continuous map $f : M \to M'$ is "(S-)definable" as shorthand for "a morphism of (S-)definable topological spaces".

We will ultimately be interested in definable structures on the topological spaces underlying comples analytic varieties, and all of the examples below are of this sort. Throughout we use the identification $\mathbb{C} \cong \mathbb{R}^2$ to speak about definable subsets of $\mathbb{C}^n$.

Example 2.3.2. (See Figure 3.) Let $\mathbb{C}^* \subset \mathbb{C}$ be the punctured plane and $e : \mathbb{C} \to \mathbb{C}^*$ the usual covering map $e(z) = e^{2\pi i z}$. We can endow $\mathbb{C}^*$ with a number of $\mathbb{R}_{\mathrm{alg}}$-definable structures:

(1) $\mathbb{C}^*$ is a (real) algebraic subset of $\mathbb{C}$, and we call this $\mathbb{R}_{\mathrm{alg}}$-definable topological space $\mathbb{G}_m^{\mathrm{def}}$.

(2) For $a \in \mathbb{R}$, define the following slope a "slanted strip" fundamental set for the covering action on $\mathbb{C}$:
$$F_a = \{z \in \mathbb{C} \mid a \cdot \operatorname{Im} z < \operatorname{Re} z < (1+\epsilon) + a \cdot \operatorname{Im} z\}.$$
F_a is evidently semialgebraic, and thus has a natural $\mathbb{R}_{\mathrm{alg}}$-definable structure. A slightly thinner open strip will inject into $\mathbb{C}^*$, and taking translates of such a strip will then give a $\mathbb{R}_{\mathrm{alg}}$-definable atlas of $\mathbb{C}^*$. We call the resulting $\mathbb{R}_{\mathrm{alg}}$-definable topological space $\mathbb{C}^*_a$. By definition the map $e : F_a \to \mathbb{C}^*_a$ is a morphism of $\mathbb{R}_{\mathrm{alg}}$-definable topological spaces.

Evidently if S, S' are two structures with $S \subset S'$ and M is an S-definable topological space, then we have an induced structure as an S'-definable space. In particular, an $\mathbb{R}_{\mathrm{alg}}$-definable structure on M will induce an S-definable structure on M for any S.

Example 2.3.3. (See Figure 3.) Consider again the previous example.

(1) The spaces $\mathbb{C}^*_a$ are all isomorphic as $\mathbb{R}_{\mathrm{alg}}$-definable topological spaces, as for instance the map $x + iy \mapsto (x + ay) + iy$ yields an isomorphism $\mathbb{C}^*_0 \xrightarrow{\cong} \mathbb{C}^*_a$.

(2) The identity map $\mathbb{C}^*_a \to \mathbb{G}_m^{\mathrm{def}}$ is *not* definable for any $a \neq 0$ in any o-minimal structure. Indeed, any ray is definable in $\mathbb{G}_m^{\mathrm{def}}$, but the preimage in F_a has infinitely many components for $a \neq 0$.

(3) The identity map $\mathbb{C}^*_0 \to \mathbb{G}_m^{\mathrm{def}}$ is not $\mathbb{R}_{\mathrm{alg}}$-definable. This is equivalent to $e : F_0 \to \mathbb{G}_m^{\mathrm{def}}$ being definable, which would imply that the real and imaginary parts $e^{-2\pi y}\cos(2\pi x)$ and $e^{-2\pi y}\sin(2\pi x)$ are $\mathbb{R}_{\mathrm{alg}}$-definable as functions $[0,1] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, which is clearly false. In fact, they are not even $\mathbb{R}_{\mathrm{an}}$-definable, as otherwise $e^{2\pi y}$ would be $\mathbb{R}_{\mathrm{an}}$-definable, whereas one can show that any $\mathbb{R}_{\mathrm{an}}$-definable function has sub-exponential growth. It is however clearly $\mathbb{R}_{\mathrm{an,exp}}$-definable (and in fact an isomorphism of $\mathbb{R}_{\mathrm{an,exp}}$-definable spaces).

Thus, of the "slanted strip" fundamental domains considered in Examples 2.3.2 and 2.3.3, the vertical strip is the unique one for which the covering map $e : F_0 \to \mathbb{G}_m^{\mathrm{def}}$ is definable in an o-minimal structure.

Remark 2.3.4. While the $\mathbb{C}^*_a$ of Example 2.3.2 are all isomorphic as $\mathbb{R}_{\mathrm{alg}}$-definable spaces, $\mathbb{C}^*_a$ and $\mathbb{C}^*_b$ do not admit a *holomorphic* isomorphism as S-definable spaces for $a \neq b$ and any o-minimal structure S. Indeed, the only holomorphic automorphisms of $\mathbb{C}^*$ are q and q^{-1} up to scaling, and one can manually check that these do not give definable isomorphisms $\mathbb{C}^*_a \to \mathbb{C}^*_b$ for $a \neq b$. However, the identity $\mathbb{C}^*_0 \to \mathbb{G}_m^{\mathrm{def}}$ does give a holomorphic $\mathbb{R}_{\mathrm{an,exp}}$-definable isomorphism.

Example 2.3.5. Let X be a real algebraic variety. Then the set of real points $X(\mathbb{R})$ equipped with the euclidean topology carries a canonical isomorphism class of $\mathbb{R}_{\mathrm{alg}}$-definable topological space structures, by covering by (finitely many) affine varieties. It is an easy exercise to see that any two (finite) affine coverings specify isomorphic $\mathbb{R}_{\mathrm{alg}}$-definable structures.

Likewise, as the complex points of an affine complex algebraic variety are naturally the real points of an affine real algebraic variety (by Weil restriction), for X

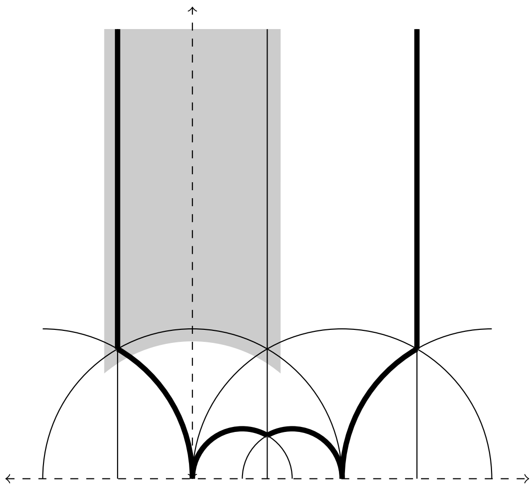


FIGURE 4. The definable fundamental set for $Y(2)$ considered in Example 2.3.7.

a complex algebraic variety the same construction yields a canonical (unique up to isomorphism) $\mathbb{R}_{\text{alg}}$-definable topological space structure on the set of complex points $X(\mathbb{C})$ with the euclidean topology.

Given a complex algebraic variety X, we define X^{eucl} to be $X(\mathbb{C})$ endowed with its euclidean topology.

Definition 2.3.6. Let X a complex algebraic variety. We define X^{def} to be the (S-)definable topological space with underlying topological space X^{eucl} and the definable structure induced from the $\mathbb{R}_{\text{alg}}$-definable structure constructed in Example 2.3.5. We refer to X^{def} as the (S-)definabilization of X.

Note that the notation does not reflect the dependence of X^{def} on the structure S.

Let (AlgVar/$\mathbb{C}$) be the category of complex algebraic varieties. It is not hard to see that we in fact have a "definabilization" functor

$$(\text{AlgVar}/\mathbb{C}) \to (S\text{-Top}) : X \mapsto X^{\text{def}}.$$

Likewise for real algebraic varieties.

Let (Top) be the category of topological spaces. Every definable space has an underlying topological space, and we denote the resulting forgetful functor

$$(S\text{-Top}) \to (\text{Top}) : X \mapsto X^{\text{top}}.$$

We then clearly have a diagram:

$$\begin{array}{ccc} (\text{AlgVar}/\mathbb{C}) & \xrightarrow{(-)^{\text{def}}} & (S\text{-Top}) \\ & {\scriptstyle (-)^{\text{eucl}}} \searrow \quad \swarrow {\scriptstyle (-)^{\text{top}}} & \\ & (\text{Top}) & \end{array}$$

There is a likewise a similar picture over $\mathbb{R}$, but for us complex algebraic varieties will play a particularly important role.

Example 2.3.7. We have the following hyperbolic analog of Examples 2.3.2 and 2.3.3. Let $Y(2)$ be the full-level two modular curve, with analytic uniformization $Y(2)^{\mathrm{an}} := \Gamma(2)\backslash\mathbb{H}$ where

$$\Gamma(2) = \left\{ A \in \mathrm{PSL}_2(\mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \right\}.$$

A fundamental domain F for the action of $\Gamma(2)$ on $\mathbb{H}$ is shown in Figure 4, corresponding to a choice of section of the quotient $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{F}_2)$. Let

$$F := \left\{ z \in \mathbb{C} \;\middle|\; |\operatorname{Re} z| < \frac{1}{2} + \epsilon \text{ and } |z|^2 > 1 - \epsilon \right\}$$

be a slight enlargement of the usual fundamental domain for the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$. Clearly F is real semialgebraic and injects into $Y(2)^{\mathrm{an}}$. The translates of F under the choosen lifts provide a cover of $Y(2)^{\mathrm{an}}$, and as the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is algebraic, this is a (finite) cover by real semialgebraic sets with real semialgebraic transition functions. Thus, we have a $\mathbb{R}_{\mathrm{alg}}$-definable structure on $Y(2)^{\mathrm{an}}$ which we call $\mathcal{Y}(2)$.

The $\mathbb{R}_{\mathrm{alg}}$-definable spaces $\mathcal{Y}(2)$ and $Y(2)^{\mathrm{def}}$ are not isomorphic via a holomorphic map, and in fact, the induced $\mathbb{R}_{\mathrm{an}}$- and $\mathbb{R}_{\mathrm{exp}}$-definable structures are not even the same, just as in Remark 2.3.4. Indeed, the image of the horoball

$$\{z \in \mathbb{H} \mid \operatorname{Im} z > 1\}$$

gives a neighborhood of the cusp at ∞ holomorphically isomorphic to Δ^*. On the one hand, in $Y(2)^{\mathrm{def}}$ there's an algebraic coordinate at the cusp which is $\mathbb{R}_{\mathrm{alg}}$-definable, and which moreover extends holomorphically to the cusp. Thus, after shrinking Δ^*, the $\mathbb{R}_{\mathrm{an}}$-definable structure induced by $Y(2)^{\mathrm{def}}$ is that of $\Delta^* \subset \mathbb{G}_m^{\mathrm{def}}$. On the other hand, the $\mathbb{R}_{\mathrm{an}}$-definable structure induced by $\mathcal{Y}(2)$ is clearly $\Delta^* \subset \mathbb{C}_0^*$.

The two structures on $Y(2)^{\mathrm{an}}$ *are* isomorphic over $\mathbb{R}_{\mathrm{an,exp}}$. Indeed, by the previous example they are isomorphic in the cuspidal neighborhoods, whereas on the complement of the union of (slightly shrunken) cuspidal neighborhoods the two structures are clearly isomorphic over $\mathbb{R}_{\mathrm{an}}$.

Remark 2.3.8. We can alternatively think of Example 2.3.7 (or indeed any of the above examples) in the following way. Let F' be an open semialgebraic fundamental set for the action of $\Gamma(2)$. The action of $\Gamma(2)$ on $\mathbb{H}$ induces a closed étale equivalence relation $R \subset \mathbb{H} \times \mathbb{H}$. Each component of this equivalence relation is evidently algebraic, and only finitely many components intersect $F' \times F'$. Thus, the restriction of the equivalence relation to F' is $\mathbb{R}_{\mathrm{alg}}$-definable. One can show that quotients by closed étale definable equivalence relations exist in the category of definable topological spaces [BBT18].

Example 2.3.9. Let X be a smooth proper complex algebraic variety. Then we may cover X^{eucl} by finitely many polydisks Δ^n. Endow each Δ^n with the $\mathbb{R}_{\mathrm{an}}$-definable structure coming from that of Δ in $(\mathbb{A}^1)^{\mathrm{def}}$. After shrinking the disks slightly the transition functions are evidently restricted analytic and therefore $\mathbb{R}_{\mathrm{an}}$-definable. This atlas gives a $\mathbb{R}_{\mathrm{an}}$-definable structure to X^{eucl} which is evidently X^{def} (over $\mathbb{R}_{\mathrm{an}}$).

Likewise, if X is a smooth complex algebraic variety (not necessarily proper), then let $\overline{X}$ be a log smooth algebraic compactification. $\overline{X}^{\mathrm{eucl}}$ can be covered by finitely many polydisks Δ^n whose intersection with X^{eucl} is of the form $(\Delta^*)^r \times \Delta^s$. This atlas then gives a $\mathbb{R}_{\mathrm{an}}$-definable structure to X^{eucl}, which is once again isomorphic to X^{def}.

Remark 2.3.10. The cylindrical cells of section 2.2 depend on an embedding into $\mathbb{R}^n$, but there is a notion of cell decomposition for definable topological spaces for which the analogs of Corollaries 2.2.9 and 2.2.10 hold. See [BBT18] for details.

3. Algebraization theorems in o-minimal geometry

O-minimal geometry has found a number of applications to the functional transcendence theory of uniformizations of algebraic varieties because it allows one to ascend and descend algebraic structures along the uniformizing map by way of two important algebraization theorems.

3.1. **The counting theorem of Pila–Wilkie.**

Definition 3.1.1. The (archimedean) *height* $H(r)$ of a rational number $r \in \mathbb{Q}$ is defined to be $\max(|a|, |b|)$, where $r = a/b$ for coprime integers a, b. Likewise, for $\alpha \in \mathbb{Q}^n$ we define the height to be $H(\alpha) = \max H(\alpha_i)$.

Note that there are finitely many points of $\mathbb{Q}^n$ of bounded height. Let $U \subset \mathbb{R}^n$ be a subset. We define the counting function as

$$N(U, t) := \# \{\alpha \in U \cap \mathbb{Q}^n \mid H(\alpha) \leq t\}$$

Furthermore, we define the algebraic and transcendental parts

$$U^{\text{alg}} := \bigcup_{\substack{Z \text{ connected semi-algebraic} \\ \dim Z > 0 \\ Z \subset U}} Z$$

$$U^{\text{tr}} := U \setminus U^{\text{alg}}.$$

Note that U^{alg} may well *not* be definable in any o-minimal structure even if U is.

The counting theorem says that rational points can only accumulate along the algebraic part in a precise sense:

Theorem 3.1.2 (Counting theorem, Theorem 1.8 of [PW06])**.** *Let $U \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then for any $\epsilon > 0$,*

$$N(U^{\text{tr}}, t) = O(t^{\epsilon}).$$

Remark 3.1.3.

(1) The o-minimal hypothesis is essential: the graph $U \subset \mathbb{R}^2$ of $\sin(\pi x)$ contains polynomially many integer points.
(2) The general form of 3.1.2 builds on an earlier result of Bombieri–Pila [BP89], which asserts the conclusion of the theorem for $U = C$ a compact real-analytic curve $C \subset \mathbb{R}^2$ containing no semi-algebraic curves, which is obviously $\mathbb{R}_{\text{an}}$-definable.
(3) There is a stronger form of 3.1.2 which is useful for applications. Informally, it states that for any $\epsilon > 0$ you can cover all the points of height at most t by at most $O(t^{\epsilon})$ semi-algebraic sets. In fact, it is this version which most naturally comes up in the proof of the Ax–Lindemann–Weierstrass and Ax–Schanuel theorems, as it is more naturally fits into inductive arguments.

    Formally speaking, it says that for any $\epsilon > 0$ there is a finite number $J = J(U, \epsilon)$ of definable sets $W^{(i)} \subset \mathbb{R}^n \times \mathbb{R}^{m_i}$ such that each fiber $W^{(i)}_y \subset \mathbb{R}^n$ is semi-algebraic and contained inside U, and a constant $c(U, \epsilon)$, such that all the rational points in U of height at most t are contained inside ct^ϵ many sets of the form $W_y^{(i)}$. See [PW06] for more details, refinements, and generalizations.

The counting theorem is often used to deduce from the presence of many rational points on U the existence of a *semialgebraic* subset $Z \subset U$ with many rational points, and this is why Theorem 3.1.2 is so powerful a tool in proving transcendence results. We will specifically need the following corollary of the strong form of Theorem 3.1.2 alluded to in the above remark:

Corollary 3.1.4. *If $N(U,t) \neq O(t^\epsilon)$ for some $\epsilon > 0$, then for any $N \in \mathbb{N}$ there is a semialgebraic subset $Z_N \subset U$ containing N rational points.*

We refer to [Pil14b] for a nice survey of the counting theorem and it's applications, but we say a few words about its role in the Pila–Zannier strategy to prove André–Oort type problems. Theorem 3.1.2 is used in two fundamentally different ways:

Let $\pi : \mathbb{H}_g \to A_g$ be the uniformizing map, $\pi_F : F \to A_g$ its restriction to a definable fundamental set, $V \subset A_g$ an algebraic subvariety, and $V_{\mathrm{sp}} \subset V$ the set of special points on V. As any subvariety with a Zariski dense set of special points is defined over a number field K, we may assume this is true for V, and thus V_{sp} is closed under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. One has to show using arithmetic arguments that special points have Galois orbits which are 'large', so that $\pi_F^{-1}(V_{\mathrm{sp}}) \subset \pi_F^{-1}(V)$ has many rational points in the sense of Theorem 3.1.2[4]. Applying Theorem 3.1.2 shows that $\pi^{-1}(V) \supset V'$ for some semi-algebraic subvariety $V' \subset \mathbb{H}_g$. Next, by Corollary 1.2.13—whose proof also uses Theorem 3.1.2 as we'll see—it then follows there is a bialgebraic $L \subset \mathbb{H}_g$ such that $\pi^{-1}(V) \supset L \supset V'$. In particular, there are special subvarieties of V containing 'most' points of V_{sp}. To finish, one has to apply an induction argument wherein special varieties are parametrized by special points on a lower-dimensional Shimura variety.

3.2. **The definable Chow theorem of Peterzil–Starchenko.**

For X a complex algebraic variety, denote by X^{an} the complex points $X(\mathbb{C})$ with its natural structure of a complex analytic variety. Recall that Chow's theorem states that if X is a proper complex variety and $Y \subset X^{\mathrm{an}}$ is a closed complex analytic subvariety, then Y is algebraic. If the properness hypothesis on X is dropped, then the theorem is false: consider for example the graph of the complex exponential in $\mathbb{C} \times \mathbb{C}^*$.

The "definable Chow" theorem of Peterzil–Starchenko essentially states that the conclusion of Chow's theorem in the non-proper case holds if Y is additionally required to be definable with respect to an o-minimal structure.

Theorem 3.2.1 (Definable Chow, Theorem 5.1 of [PS03])**.** *Fix an o-minimal structure and let X be a complex algebraic variety. Then any closed complex analytic subvariety $Y \subset X^{\mathrm{an}}$ whose underlying set is definable in X^{def} is algebraic.*

Note that it is enough to assume Y is (analytically) irreducible of dimension d. Furthermore, we may replace X with a (nonempty) affine Zariski open subset U and algebraize $U^{\mathrm{an}} \cap Y$, for then Y is the closure of $U^{\mathrm{an}} \cap Y$. We can thus assume $X = \mathbb{A}^n$, and in the sequel we'll simply write $\mathbb{C}^n = (\mathbb{A}^n)^{\mathrm{an}}$.

[4]This is classical in the case of the modular curve, much harder for A_g, and still open in general. See [Tsi18]

We'll give two proofs, the first of which minimizes the explicit use of o-minimality, and the second that of complex analysis. The first proof relies on an important analyticity criterion of Bishop:

Theorem 3.2.2 (Theorem 3 of [Bis64]). *Let $U \subset \mathbb{C}^n$ be an open subset and $Z \subset U$ a closed analytic subset. If $Y \subset U \backslash Z$ is a pure dimension d closed analytic subset of finite $2d$-dimensional volume, then the closure $\overline{Y}$ of Y in U is an analytic subset.*

First proof of Theorem 3.2.1. Consider $\mathbb{C}^n \subset \mathbb{P}^n$ with complement $\mathbb{P}^{n-1}$ the plane at infinity. By the lemma below, Y has finite volume locally around $\mathbb{P}^{n-1}$, so by Bishop's theorem the closure $\overline{Y}$ of Y in $\mathbb{P}^n$ is an analytic subvariety, hence algebraic by the usual Chow theorem.

Lemma 3.2.3. Any bounded k-dimensional definable $V \subset \mathbb{R}^m$ has finite k-volume.

Proof. A bounded k-dimensional definable subset of $\mathbb{R}^k$ certainly has finite volume. The volume of $V \subset \mathbb{R}^m$ is bounded up to a constant by the maximum volume of its coordinate projections to $\mathbb{R}^k$—which is finite—times the maximum degree of these projections, which is also finite. □

□

The second proof relies on the following fact using only elementary complex analysis.

Lemma 3.2.4. Any definable holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is algebraic.

Proof.

Step 1. An entire definable function $f : \mathbb{C} \to \mathbb{C}$ is algebraic.

f cannot have an essential singularity at infinity or else it would have infinite fibers, by Casorati–Weierstrass.

Step 2. Any definable holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is algebraic.

Write $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$. For any $w \in \mathbb{C}^{n-1}$, $f(z, w)$ is a polynomial in z by Step 1. By Corollary 2.2.10, the degree of $f(z, w)$ in z is uniformly bounded[5], so for some N,

$$f(z, w) = \sum_{k=0}^{N} \frac{\partial^k f}{\partial z^k}(0, w) \frac{z^k}{k!}.$$

By induction (using the previous step as the base case) the definable holomorphic functions $\frac{\partial^k f}{\partial z^k}(0, w) : \mathbb{C}^{n-1} \to \mathbb{C}$ are algebraic.

□

Second proof of Theorem 3.2.1. We prove the claim by induction on the dimension d of Y, the base case being obvious.

Step 1. The boundary $\partial Y := \overline{Y} \backslash Y \subset \mathbb{P}^{n-1}$ of Y in $\mathbb{P}^n$ is a definable subset of (real) dimension at most $2d - 1$.

From cell decomposition, the boundary of a definable set always has smaller dimension.

[5] We might have to consider $f(z, w) - c$ to avoid multiplicity.

Step 2. There is a linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$ for which the restriction $\pi_Y : Y \to \mathbb{C}^d$ is proper.

Linear projections $\mathbb{C}^n \to \mathbb{C}^{n-1}$ are obtained by projecting from a point $p \in \mathbb{P}^{n-1}$ at infinity; the fibers of this projection are the lines through p (minus the point p itself). As $d < n$, by the previous step $\dim_{\mathbb{R}} \partial Y < \dim_{\mathbb{R}} \mathbb{P}^{n-1} = 2n-2$, so there is a projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ for which each fiber has bounded intersection with Y. The projection $Y \to \mathbb{C}^{n-1}$ is therefore proper, and the image is clearly definable and closed analytic by Remmert's proper mapping theorem. Now iterate.

Step 3. The locus $Y_0 \subset Y$ where $\pi_Y : Y \to \mathbb{C}^d$ is not étale is a closed algebraic subvariety Y_0 of $\mathbb{C}^n$.

Y_0 is analytic of strictly smaller dimension than Y and evidently definable (as for instance it is the locus where the fiber size is nongeneric). By the inductive hypothesis we therefore have that Y_0 is algebraic.

Step 4. Y is algebraic.

Write $\mathbb{C}^n = \mathbb{C}^{n-d} \times \mathbb{C}^d$, so π is projection to the second factor. Let $Z^{\mathrm{an}} = \pi(Y_0)$, which is a closed algebraic subvariety of $\mathbb{C}^d$. Let N be the degree of the map $\pi_Y : Y \to \mathbb{C}^d$ and consider the function

$$F : \mathbb{C}^d \backslash Z^{\mathrm{an}} \to \mathrm{Sym}^N \mathbb{C}^{n-d} : z \mapsto \pi^{-1}(z).$$

Note that $\mathrm{Sym}^N \mathbb{C}^{n-d}$ is an affine algebraic variety. F is evidently definable and holomorphic, as well as locally bounded around Z^{an} (as π_Y is proper). Thus, the pullbacks of the coordinate functions of $\mathrm{Sym}^N \mathbb{C}^{n-d}$ extend to definable holomorphic functions $f : \mathbb{C}^d \to \mathbb{C}$, which are therefore algebraic by Lemma 3.2.4. It follows that $Y \backslash Z^{\mathrm{an}}$ is algebraic, and therefore that Y is. □

Remark 3.2.5. Neither of these proofs is the one given by Peterzil–Starchenko—as they prove it for arbitrary real closed fields—but the second proof is close to that of [PS03]: we've only really cheated by using Casorati–Weierstrass. Step 1 of the proof of Lemma 3.2.4 can be proven in general using a version of Liouville's theorem proven by Peterzil–Starchenko.

4. The Ax–Lindemann–Weierstrass theorem

In this section, as a warm up for the proof of Theorem 6.1.1, we show how to use the Pila–Wilkie theorem to prove the Ax–Lindemann–Weierstrass theorem for the exponential map. Many of the same arguments will be used in the proof of Theorem 6.1.1. The notable exception is that the definable Chow theorem does not play a role in the proof of the Ax–Lindemann–Weierstrass theorem but is essential to the proof of the Ax–Schanuel theorem.

4.1. **The exponential function.**

Let

$$\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n : (z_1, \ldots, z_n) \mapsto (e(z_1), \ldots, e(z_n))$$

where $e(z) = e^{2\pi i z}$. Let's first give a proof of the classification of the bialgebraic subvarieties of $\mathbb{C}^n$ which only mildly uses some of the o-minimal machinery—and in particular will not use either of the algebraization theorems discussed in the previous lecture.

Consider a closed irreducible algebraic subvariety $M \subset (\mathbb{C}^*)^n$ and the induced map on fundamental groups

$$\pi_1(M) \to \pi_1((\mathbb{C}^*)^n) \cong \mathbb{Z}^n.$$

The important observation is that we can directly relate the size of the monodromy (that is, the image of $\pi_1(M)$) to the invariance of M.

Proposition 4.1.1. *If the image of $\pi_1(M)$ is not finite index in $\pi_1((\mathbb{C}^*)^n)$, then M is contained in a coset of a proper algebraic sub-torus.*

Proof. Without loss of generality we may assume

$$\pi_1(M) \to 0 \oplus \mathbb{Z}^{n-1} \subset \mathbb{Z}^n.$$

Let $\tilde{M}$ be a connected component of $\pi^{-1}(M)$. The condition on the monodromy means $z_1|_{\tilde{M}}$ is invariant under $\pi_1(M)$. Hence z_1 descends to a holomorphic function on M. We claim that z_1 has bounded real part, and this will imply it is constant.

For the claim it is sufficient to find a fundamental set Σ for $\tilde{M} \to M$ such that z_1 has bounded real part on Σ. Let

$$F = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid -\epsilon < \mathrm{Re}(z_i) < 1 + \epsilon\} \tag{6}$$

be the usual fundamental set for $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$; clearly $z_1|_F$ has bounded real part. There is a finite cover M_i of M by simply-connected open subsets each of which therefore lifts to F. We may then obtain Σ as a union of translates of these lifts, and the claim is proved.

□

Corollary 4.1.2. *The closed irreducible bialgebraic subvarieties of $(\mathbb{C}^*)^n$ are precisely cosets of algebraic sub-tori.*

Proof. Equivalently, we must show that the closed irreducible bialgebraic subvarieties of $\mathbb{C}^n$ are translates of $\mathbb{C}$-subspaces defined over $\mathbb{Q}$. We may first assume that L is not stabilized by a nonzero integral translation. Indeed, as the stabilizer of L is an algebraic subgroup of the vector group $\mathbb{C}^n$, if it is stabilized by $\Lambda \subset \mathbb{Z}^n$ then it is stabilized by $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. We may then quotient out by the torus generated by Λ. Second, by the proposition we may assume that the monodromy of $\pi(L)$ is finite-index in $\mathbb{Z}^n$, by passing to a subtorus.

With these assumptions, the following claim then shows that $n = 0$, completing the proof.

Claim. If $M := \pi(L)$ has infinite monodromy, then L is stabilized by an integral translation.

Note that L is an irreducible component of $\pi^{-1}(M)$. In the notation of the proof of Proposition 4.1.1, let $\pi_F := \pi|_F$ be the restriction to the standard fundamental domain. Some coordinate of L has unbounded real part, and as L is tiled by translates of $\pi_F^{-1}(M)$ the set

$$I := \{v \in \mathbb{R}^n \mid \dim((L + v) \cap \pi_F^{-1}(M)) = \dim L\}$$

contains an infinite set of integral points. As I is definable in an o-minimal structure, there is a (definable) connected continuous path $\{v_t\} \subset I$ connecting two distinct integral points $v_0, v_1 \in \mathbb{Z}^n$ of I. It follows that $L + v_t \subset \pi^{-1}(M)$ for all t, but as $L + v_0$ is an irreducible component we must have $L + v_0 = L + v_t$ for all t and in particular $v_1 - v_0$ stabilizes L, whence the claim. □

We are now ready to prove the Ax–Lindemann–Weierstrass theorem, whose statement we recall.

Theorem 4.1.3 (Ax–Lindemann–Weierstrass)**.** *Suppose there are algebraic subvarieties $V_1 \subset \mathbb{C}^n$ and $V_2 \subset (\mathbb{C}^*)^n$.*

(1) If $\pi(V_1) \subset V_2$, then there is a bialgebraic $M \subset (\mathbb{C}^*)^n$ with

$$\pi(V_1) \subset M \subset V_2;$$

(2) If $\pi(V_1) \supset V_2$, then there is a bialgebraic $M \subset (\mathbb{C}^*)^n$ with

$$\pi(V_1) \supset M \supset V_2.$$

Before the proof we make a crucial observation: both the fundamental set $F \subset \mathbb{C}^n$ and the restriction $\pi_F : F \to (\mathbb{C}^*)^n$ of the covering map are definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ (*c.f.* Example 2.3.3).

Proof of Theorem 4.1.3. We start with the proof of (1). We can assume by taking closures and components that V_1 (resp. V_2) is a closed irreducible algebraic subvariety of $\mathbb{C}^n$ (resp. $(\mathbb{C}^*)^n$). We can further assume that V_2 is not contained in any proper subtorus, and that V_1 is a maximal closed irreducible algebraic subvariety of $\pi^{-1}(V_2)$. It remains to show that V_1 is bialgebraic.

Consider the set

$$I := \{v \in \mathbb{R}^n \mid \dim\left((V_1 + v) \cap \pi_F^{-1}(V_2)\right) = \dim V_1\}.$$

As V_1 is irreducible, we see that $v \in I$ if and only if the translate $V_1 + v$ meets F and $V_1 + v \subset \pi^{-1}(V_2)$.

Step 1. I is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Indeed, the universal translate

$$\mathcal{V}_1 := \{(v, z) \mid z \in V_1 + v\} \subset \mathbb{R}^n \times \mathbb{C}^n$$

is (real) algebraic so definable, as therefore is the universal intersection

$$\mathcal{U} := \mathcal{V}_1 \cap \left(\mathbb{R}^n \times \pi_F^{-1}(V_2)\right).$$

Applying Corollary 2.2.10 to the projection $\mathcal{U} \to \mathbb{R}^n$ yields the claim.

Step 2. $\mathrm{Stab}_{\mathbb{Z}^n}(V_1)$ is infinite.

We may assume V_1 meets F, as $\pi^{-1}(V_2)$ is covered by integral translates[6] of F. Note that for any $v \in \mathbb{Z}^n$, V_1 meets $F - v$ if and only if $v \in I$, so the integral points of I correspond to fundamental domains that V_1 passes through. Observe that V_1 cannot be contained in any "height ball"

$$\bigcup_{\substack{v \in \mathbb{Z}^n \\ H(v) \le r}} (F - v)$$

as then each coordinate z_i would have bounded real part and therefore be constant. For each $t \in \mathbb{Z}_{>0}$, the complement of the "height sphere"

$$\bigcup_{\substack{v \in \mathbb{Z}^n \\ H(v) = t}} (F - v)$$

[6]Strictly speaking we should take $\epsilon = 0$ and make F a fundamental *domain* for this argument.

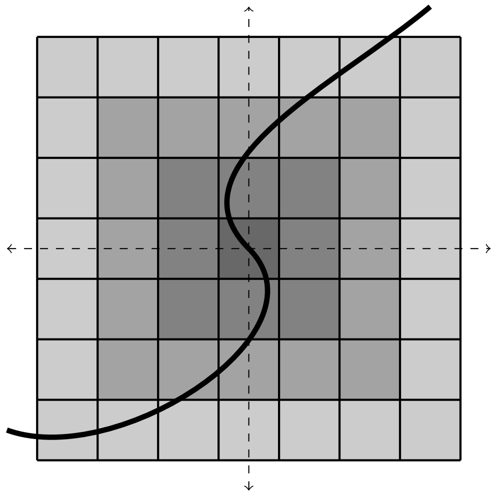


FIGURE 5. V_1 must pass through at least one fundamental domain $F - v$ of each height.

has two connected components, so V_1 must pass through it (see Figure 5). Thus, we have

$$N(I, t) \geq t + 1.$$

By the strong form of the counting theorem, we have a (real) semialgebraic curve $C \subset I$ that contains at least two integral points.

If translation by $c \in C$ does not stabilize V_1, then $\bigcup_c (V_1 + c)$ is a real semialgebraic subset of $\pi^{-1}(V_2)$, and its $\mathbb{C}$-Zariski closure is a complex algebraic subvariety of $\pi^{-1}(V_2)$ of larger dimension than V_1, contradicting the maximality of V_1. Thus, $V_1 = V_1 + c$ for all $c \in C$, and V_1 is stabilized by a non-zero integer point.

Step 3. Induction step.

Since $\mathrm{Stab}_{\mathbb{C}^n}(V_1)$ is an algebraic subgroup, it follows from the previous step that V_1 is stabilized *by a complex line* $\mathbb{C} \subset \mathbb{C}^n$ *defined over* $\mathbb{Q}$. Thus, there is a splitting $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$ defined over $\mathbb{Q}$ such that $V_1 = V_1' \times \mathbb{C}$. Let $V_2' = V_2 \cap (\mathbb{C}^*)^{n-1}$. Since the proposition is trivially true for $n = 1$, we may inductively assume there is a bialgebraic $L' \subset \mathbb{C}^{n-1}$ with

$$V_1' \subset L' \subset \pi^{-1}(V_2').$$

By the assumption on V_2, we must have $L' \neq \mathbb{C}^{n-1}$ (or else $V_2 = (\mathbb{C}^*)^n$), so we can apply the induction hypothesis again to $V_2'' = \pi(L' \oplus \mathbb{C}) \cap V_2$ and $V_1'' = V_1$. We conclude there is a bialgebraic $L'' \subset L' \oplus \mathbb{C}$ with

$$V_1 \subset L'' \subset \pi^{-1}(V_2'') \subset \pi^{-1}(V_2)$$

and so $V_1 = L''$ is bialgebraic, by the maximality of V_1.

The proof of part (2) is very similar, so we just sketch the argument. We may now assume V_1 is the $\mathbb{C}$-Zariski closure of a component of $\pi^{-1}(V_2)$ and apply the Pila–Wilkie theorem to

$$I := \{v \in \mathbb{R}^n \mid \dim\big((V_1 + v) \cap \pi_F^{-1}(V_2)\big) = \dim V_2\}.$$

We can then conclude that there is a real semialgebraic $C \subset I$, and if translation by $c \in C$ doesn't stabilize V_1, then $\bigcap_c (V_1 + c)$ would contain $\pi^{-1}(V_2)$, implying that the $\mathbb{C}$-Zariski closure of $\pi^{-1}(V_2)$ is smaller than V_1, a contradiction. We conclude that V_1 is invariant under a $\mathbb{C}$-line defined over $\mathbb{Q}$, and a similar induction yields the claim.

□

4.2. **Hyperbolic uniformizations.**

We give a sketch of how the above proof is adapted to the setting of Shimura varieties, but we first recall the basic structures associated with Shimura varieties (see [Mil13] for details). These are:

- A connected semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$.
- A bounded symmetric domain

  $$\Omega = \mathbf{G}(\mathbb{R})/K$$

  where K is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. Ω is a complex manifold and its biholomorphism group is $\mathbf{G}(\mathbb{R})$. It also carries a natural left-invariant hermitian metric h which has negative sectional curvature. Note that the requirement that Ω have a holomorphic structure is a strong requirement on the group $\mathbf{G}$.
- The compact dual $\check{\Omega}$, which is

  $$\check{\Omega} = \mathbf{G}(\mathbb{C})/B$$

  where B is a maximal Borel subgroup. It is a homogeneous projective variety. The Harish-Chandra embedding theorem shows that for any choice of B containing K, Ω is realized as a semialgebraic subset of $\check{\Omega}$. Moreover, this embedding is unique up to the action of $\mathbf{G}(\mathbb{C})$.
- An arithmetic lattice $\Gamma \subset \mathbf{G}(\mathbb{Q})$, that is, a subgroup which is commensurable to the subgroup preserving an integral structure $H_{\mathbb{Z}}$ in a faithful representation $\mathbf{G}(\mathbb{Q}) \to \mathrm{GL}(H_{\mathbb{Q}})$. Γ is discrete and finite co-volume in $\mathbf{G}(\mathbb{R})$ (with respect to a left-invariant metric).
- The analytic quotient

  $$Y = \Gamma\backslash\Omega = \Gamma\backslash\mathbf{G}(\mathbb{R})/K.$$

  Y uniquely has the structure of an algebraic variety [BB66], and it is called a *Shimura variety.*

Example 4.2.1. For $\mathbf{G} = \mathrm{Sp}_{2g}$ and $\Gamma = \mathrm{Sp}_{2g}(\mathbb{Z})$ we have

$$\begin{aligned} H_{\mathbb{Z}} &= \text{the unimodular symplectic lattice of rank } 2g \\ \Omega &= \text{Siegel upper half-space } \mathbb{H}_g \\ \check{\Omega} &= \text{the Lagrangian Grassmannian of } H_{\mathbb{C}} \\ Y = \Gamma\backslash\mathbb{H}_g &= \text{the (coarse) moduli space of principally polarized} \\ &\quad\ g\text{-dimensional abelian varieties } A_g \end{aligned}$$

We can now consider the uniformization $\pi : \Omega \to Y$. Recall that we say a complex analytic subvariety $V \subset \Omega$ is algebraic if there is an algebraic subvariety $\check{V} \subset \check{\Omega}$ with $V = \check{V} \cap \Omega$. We say an algebraic subvariety $V \subset \Omega$ is bialgebraic if $\dim V = \dim \pi(V)^{\mathrm{Zar}}$, as in Definition 1.2.8. The bialgebraic subvarieties are the so-called weakly special subvarieties:

Definition 4.2.2. A *weakly special* subvariety of Y is a Shimura variety Y' given as

$$Y' = \Gamma' \backslash \mathbf{G}'(\mathbb{R})/K'$$

where $\mathbf{G}'$ is an algebraic $\mathbb{Q}$-subgroup of $\mathbf{G}$, $\Gamma' = \Gamma \cap \mathbf{G}'(\mathbb{Q})$ is an arithmetic lattice, and $K' = K \cap \mathbf{G}'(\mathbb{R})$. Evidently Y' is then an analytic subvariety of Y, and in fact it is (uniquely) an algebraic subvariety.

Proposition 4.2.3 (Theorem 1.2 of [UY11])**.** *Let Y be a Shimura variety. The closed irreducible bialgebraic subvarieties of Y are precisely the weakly special subvarieties.*

As in Proposition 4.1.1, the proof of Proposition 4.2.3 uses monodromy arguments and relies heavily on the work of André–Deligne [And92, Del71]. The Ax–Lindemann–Weierstrass theorem in this context was proven by Pila for powers of the modular curve [Pil11], by Pila–Tsimerman for A_g [PT14], and then by Klingler–Ulmo–Yafaev for general Shimura varieties [KUY16]:

Theorem 4.2.4 (Ax–Lindemann–Weierstrass, Theorem 1.6 of [KUY16])**.** *Let Y be a Shimura variety uniformized by Ω. Suppose there are algebraic subvarieties $V_1 \subset \Omega$ and $V_2 \subset Y$.*

(1) If $\pi(V_1) \subset V_2$, then there is a bialgebraic $M \subset \Omega$ with

$$\pi(V_1) \subset M \subset V_2;$$

(2) If $\pi(V_1) \supset V_2$, then there is a bialgebraic $M \subset \Omega$ with

$$\pi(V_1) \supset M \supset V_2.$$

Sketch of proof. We will only sketch the proof of (1), as (2) is similar. We can make the same assumptions on V_1 and V_2 as in the proof of Theorem 4.1.3—that is, that both are closed irreducible subvarieties. We can further assume V_2 is not contained in any bialgebraic subvariety, and that V_1 is a maximal algebraic subvariety of $\pi^{-1}(V_2)$.

We follow the same three steps as the proof of Theorem 4.1.3

Step 1.

We first need a definable fundamental set $F \subset \Omega$ for which the restriction $\pi_F : F \to Y$ is a definable quotient map. In [KUY16], this is done using finitely many Siegel sets, which yield a semialgebraic fundamental set $\tilde{F} \subset \mathbf{G}(\mathbb{R})$ for the action of any arithmetic lattice $\Gamma \subset \mathbf{G}(\mathbb{Q})$. We can then take F as the image of $\tilde{F}$ in $\Omega = \mathbf{G}(\mathbb{R})/K$. It is then shown that $\pi_F : F \to Y$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable using the theory of toroidal compactifications. In Lecture 5 we will instead use the local theory of degenerations of Hodge structures to produce a definable fundamental set.

It then follows in the same way that

$$I := \{g \in \mathbf{G}(\mathbb{R}) \mid \dim\left(gV_1 \cap \pi_F^{-1}(V_2)\right) = \dim V_1\}.$$

is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Step 2. $\mathrm{Stab}_{\mathbf{G}(\mathbb{Z})}(V_1)$ is infinite.

We would like to apply the Pila–Wilkie theorem to I as in Step 2 of the proof of Theorem 4.1.3, so we need

Claim. $N(I,t) \gg t^{\epsilon}$ for some $\epsilon > 0$.

We postpone until Lecture 7 the precise definition of the height of an element of $\mathbf{G}(\mathbb{Q})$ and the counting function. The above argument using "height balls" to produce polynomially many $\mathbb{Z}$-points of I (in the height) breaks down, essentially because the uniformizing group Γ and its action on Ω are now very complicated.

The problem is remedied in [PT14, KUY16] by instead using *metric balls.* Let Γ_V be the image of the monodromy representation $\pi_1(V_2) \to \mathbf{G}(\mathbb{Q})$. Recall that since $\pi^{-1}(V_2)$ is stable under Γ_V, it will be sufficient to show that V_1 passes through polynomially many (in the height of γ) integral translates $\gamma^{-1}F$ for $\gamma \in \Gamma_V$. We may assume V_1 meets F and take a basepoint $x_0 \in F \cap V_1$. Consider the metric balls $B_{x_0}(R)$ centered at x_0. By a result of Hwang–To, the volume achieved by V_1 in $B_{x_0}(R)$ is large:

Theorem 4.2.5 (Corollary 3 of [HT02])**.** *There is a constant $\beta > 0$ only depending on Ω such that for any closed positive-dimensional analytic subvariety $Z \subset B_{x_0}(R)$ we have*

$$\mathrm{vol}(Z) \gg \sinh(\beta R)^{\dim Z} \operatorname{mult}_{x_0} Z.$$

We will need a version of Theorem 4.2.5 for period domains, whose proof we sketch in Lecture 8.

To establish the claim, it now remains to show that:

(a) The only integral translates $\gamma^{-1}F$ meeting $B_{x_0}(R)$ have $H(\gamma) \ll e^{O(R)}$;
(b) V_1 has bounded volume intersection with all of the translates $\gamma^{-1}F$.

Indeed, the volume of $V_1 \cap B_{x_0}(R)$ is exponential in the radius by Theorem 4.2.5, so by (b) and the fact that the $\gamma^{-1}F$ cover $\pi^{-1}(V_2)$ with bounded overlaps we conclude that V_1 passes through exponentially many (in the radius) integral translates $\gamma^{-1}F$ in $B_{x_0}(R)$. It then follows from (a) that the number of these integral translates is polynomial in the height.

For (a), we need to compare the metric dilation of γ to its height, which is standard (see for example Lecture 7). For (b), it suffices to show that all translates gV_1 for $g \in \mathbf{G}(\mathbb{C})$ meet F with bounded volume, and since these translates form an algebraic family, we can use definability to get a uniform bound (see for example Proposition 5.5.1).

To finish, just as in the proof of Theorem 4.1.3, we obtain an algebraic family $\{g_c\}_{c\in C} \subset \mathbf{G}(\mathbb{C})$ with $g_cV_1 \subset \pi^{-1}(V_2)$ by applying the Pila–Wilkie theorem. If V_1 is a *maximal* irreducible algebraic subvariety of $\pi^{-1}(V_2)$, then we have $V_1 = \bigcup_{c\in C} g_cV_1$ and V_1 then is therefore invariant under $\{g_c\}_{c\in C}$ (which in particular contains a nontrivial integral point).

Step 3. Induction step.

As the stabilizer of V_1 is an algebraic subgroup of $\mathbf{G}$ and we know from the previous step that $\mathrm{Stab}_{\mathbf{G}(\mathbb{Z})}(V_1)$ is infinite, it follows that V_1 is stabilized by a positive-dimensional connected $\mathbb{Q}$-subgroup $\mathbf{H}$ of $\mathbf{G}$, namely the identity component of the $\mathbb{Q}$-Zariski closure of $\mathrm{Stab}_{\mathbf{G}(\mathbb{Z})}(V_1)$. However, to make the induction work, one needs V_1 to be stabilized by a *normal* $\mathbb{Q}$-subgroup of $\mathbf{G}$, as this will imply $\mathbf{G}$ is isogeneous to a product. This problem is solved using Hecke correspondences in [PT14, KUY16]. In [MPT17], the same problem is solved in a different way to prove the Ax–Schanuel theorem, essentially by using the definable Chow theorem to algebraize the family of algebraic deformations V_1' of V_1 that are contained in $\pi^{-1}(V_2)$, and then using the fact that algebraic families of varieties have large monodromy. We will use the same strategy in Lecture 6.

□

5. Recollections from Hodge theory

Shimura varieties are moduli spaces of very special polarized Hodge structures, and it is very natural to formulate the Ax–Schanuel conjecture (as well as the other transcendence statements) for general moduli spaces of polarized Hodge structures. We spend this lecture recalling the relevant notions from Hodge theory. We will be necessarily brief, and refer the interested reader to [CMSP03] and [GGK12] for details.

5.1. **Preliminaries.**

Definition 5.1.1. Fix an integer n. Let $H_{\mathbb{Z}}$ be a finite rank free $\mathbb{Z}$-module. A pure Hodge structure on $H_{\mathbb{Z}}$ of weight n is a decomposition into complex vector spaces

$$H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q} \tag{7}$$

satisfying $\overline{H^{p,q}} = H^{q,p}$. The dimensions $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}$ are called the Hodge numbers. We say the Hodge structure is effective if $H^{p,q} = 0$ for $p > n$.

Note that the Hodge structure is determined by the *Hodge filtration*

$$F^p := \bigoplus_{r \geq p} H^{r,s}$$

as $H^{p,q} = F^p \cap \overline{F^q}$. Conversely, a descending filtration $F^\bullet$ determines a Hodge structure of weight n if it satisfies

$$F^p \cap \overline{F^{n-p+1}} = 0 \tag{8}$$

for all p.

Example 5.1.2. A pure weight 1 (or -1) Hodge structure is equivalent to a complex torus T. We canonically have an embedding

$$H_1(T, \mathbb{Z}) \to H^0(T, \Omega^1_T)^\vee \oplus H^0(T, \overline{\Omega}^1_T)^\vee : \gamma \mapsto \int_\gamma$$

which yields a decomposition

$$H_1(T, \mathbb{C}) = H^{-1,0} \oplus H^{0,-1}$$

with $H^{-1,0} = H^0(T, \Omega^1_T)^\vee$ and $H^{0,-1} = \overline{H^{-1,0}}$. Projecting $H_1(T, \mathbb{Z})$ to $H^{-1,0}$ we can recover T canonically by the albanese

$$T \xrightarrow{\cong} H^0(T, \Omega^1_T)^\vee / H_1(T, \mathbb{Z}) : p \mapsto \int_0^p .$$

The weight -1 Hodge structure on $H_1(T, \mathbb{Z})$ naturally induces a weight 1 Hodge structure on $H^1(T, \mathbb{Z})$.

Definition 5.1.3. Suppose $H_{\mathbb{Z}}$ carries a weight n Hodge structure, and let $q_{\mathbb{Z}}$ be a $(-1)^n$-symmetric bilinear form—that is, $q_{\mathbb{Z}}$ is symmetric if n is even and skew-symmetric if n is odd.

(1) The Weil operator $C \in \mathrm{End}(H_{\mathbb{R}})$ is the real endomorphism satisfying

$$C_{\mathbb{C}} = \bigoplus_{p,q} i^{p-q} \cdot \mathrm{id}_{H^{p,q}} .$$

(2) The *Hodge form* is the hermitian form h on $H_{\mathbb{C}}$ defined by

$$h(u, v) = q_{\mathbb{C}}(Cu, \overline{v}).$$

(3) We say the Hodge structure is *polarized* by $q_\mathbb{Z}$ if the Hodge form is positive-definite and the decomposition (7) is h-orthogonal.

If the Hodge structure is polarized by $q_\mathbb{Z}$, then the Hodge filtration $F^\bullet$ is $q_\mathbb{C}$-isotropic: we have $(F^\bullet)^\perp = F^{n+1-\bullet}$. Conversely, a $q_\mathbb{C}$-isotropic Hodge filtration satisfying (8) determines a $q_\mathbb{Z}$-polarized Hodge structure if the Hodge form is positive-definite.

Example 5.1.4. A polarized weight 1 (or -1) Hodge structure is equivalent to a polarized abelian variety A. A skew-symmetric integral form $q_\mathbb{Z}$ on $H_1(A,\mathbb{Z})$ can be thought of as an element $h \in H^2(A,\mathbb{Z})$. By the Lefschetz $(1,1)$ theorem, the $q_\mathbb{C}$-isotropicity condition on the Hodge decomposition implies $h = \mathrm{c}_1(L)$ for a line bundle L on A, and the positivity condition implies L is ample.

Example 5.1.5. We have the following broad generalization of the previous example, which was the original motivation for their introduction. Let Y be a proper Kähler manifold (for example a smooth complex projective variety). After choosing a Kähler form ω, we obtain a weight n Hodge structure on degree n singular cohomology

$$H^n(Y,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(Y) \tag{9}$$

by decomposing harmonic representatives of de Rham cohomology classes into (p,q) parts. Furthermore, suppose Y is a smooth complex projective variety with ample bundle L and set $h = \mathrm{c}_1(L)$. The singular cohomology $H^*(Y,\mathbb{Q})$ decomposes into polarized Hodge structures as follows. For $n \le d = \dim X$, let

$$H^{d-n}_{\mathrm{prim}}(Y,\mathbb{Z}) := \ker\left(h^{n+1}\cup : H^{d-n}(Y,\mathbb{Z})_{\mathrm{tf}} \to H^{d+n+2}(Y,\mathbb{Z})_{\mathrm{tf}}\right).$$

Where $(-)_{\mathrm{tf}}$ denotes the torsion-free quotient. We have

$$H^n(Y,\mathbb{Q}) = \bigoplus_{0\le k\le n/2} h^k \cup H^{n-2k}_{\mathrm{prim}}(Y,\mathbb{Q}).$$

$H^n_{\mathrm{prim}}(Y,\mathbb{Z})$ carries a natural integral form

$$q_n(a,b) := \int_Y h^{\dim Y - 2n} \cup a \cup b.$$

The decomposition (9) (associated to the Kähler class h) then induces a weight n Hodge structure on $H_{\mathrm{prim}}(Y,\mathbb{Z})$ polarized by q_n.

Remark 5.1.6. Note that if $H_\mathbb{Z}$ carries a pure Hodge structure, then so too will any tensor power, symmetric power, wedge power, etc. of $H_\mathbb{Z}$. The same is true of pure polarized Hodge structures.

5.2. **Period domains and period maps.**

Define the algebraic $\mathbb{Q}$-group $\mathbf{G}(\mathbb{Q}) = \mathrm{Aut}(H_\mathbb{Q}, q_\mathbb{Q})$; we will often denote $\mathbf{G}(\mathbb{Z}) = \mathrm{Aut}(H_\mathbb{Z}, q_\mathbb{Z})$. It is then not hard to see that the space D of $q_\mathbb{Z}$-polarized pure weight n Hodge structures on $H_\mathbb{Z}$ with specified Hodge numbers $h^{p,q}$ is a homogeneous space for $\mathbf{G}(\mathbb{R})$. Indeed, choosing a reference Hodge structure, we have

$$D = \mathbf{G}(\mathbb{R})/V$$

where V is a subgroup of the compact unitary subgroup $K = \mathbf{G}(\mathbb{R}) \cap U(h)$ of $\mathbf{G}(\mathbb{R})$ with respect to the hodge form of the reference Hodge structure. Moreover, D is canonically an open subset (in the euclidean topology) of $\check{D} = \mathbf{G}(\mathbb{C})/P$, the flag variety parametrizing $q_{\mathbb{C}}$-isotropic Hodge filtrations $F^\bullet$ on $H_{\mathbb{C}}$ with $h^{p,n-p} = \dim F^p/F^{p+1}$.

Definition 5.2.1. Such a D is called a *polarized period domain.*

Example 5.2.2. Given a smooth projective morphism $f : Y \to X$, consider the local system $R^k f_*\mathbb{Q}$ for some k. In the notation of Example 5.1.5, $R^n f_*\mathbb{Z}$ can be decomposed into primitive pieces, and each fiber of $R^n_{\mathrm{prim}} f_*\mathbb{Z}$ carries a pure weight n Hodge structure. By a theorem of Griffiths, the resulting map

$$\varphi : X^{\mathrm{an}} \to \mathbf{G}(\mathbb{Z})\backslash D : y \mapsto [H^n_{\mathrm{prim}}(X_y, \mathbb{Z})]$$

is holomorphic and locally liftable to D.

The fundamental observation of Griffiths is that we cannot get arbitrary maps to $\mathbf{G}(\mathbb{Z})\backslash D$ from geometry as in Example 5.2.2. Indeed, only certain tangent directions of D are accessible to algebraic families. To make this precise, fix a point $x \in D$ and note that a deformation of the Hodge filtration at x in particular yields a deformation of each F^p_x, so we have a natural map

$$T_x D \to \bigoplus_p \mathrm{Hom}(F^p_x, H_{\mathbb{C}}/F^p_x) \tag{10}$$

Definition 5.2.3. The Griffiths transverse subspace $T^{GT}_x D \subset T_x D$ is the inverse image of $\bigoplus_p \mathrm{Hom}(F^p_x, F^{p-1}/F^p_x)$ under the map in (10).

In other words, to first order each F^p is only deformed inside F^{p-1}. The Griffiths transverse subspaces assemble into a holomorphic subbundle $T^{GT} D \subset TD$.

Remark 5.2.4. Each pure polarized Hodge structure $x \in D$ on $H_{\mathbb{Z}}$ naturally induces a pure polarized Hodge structure on the Lie algebra $\mathfrak{g}_{\mathbb{R}} \subset \mathrm{End}(H_{\mathbb{R}})$ of weight 0, which we call $\mathfrak{g}_x$. Denote its Hodge filtration by $F^\bullet_x \mathfrak{g}_{\mathbb{C}}$. The Lie algebra of the stabilizer $P_x \subset \mathbf{G}(\mathbb{C})$ of $x \in \check{D}$ is then naturally $F^0_x \mathfrak{g}_{\mathbb{C}}$. Thus, the tangent space $T_x D$ is naturally (and holomorphically) identified with $\mathfrak{g}_{\mathbb{C}}/F^0_x \mathfrak{g}_{\mathbb{C}}$. The Griffiths transverse subspace is $F^{-1}_x \mathfrak{g}_{\mathbb{C}}/F^0_x \mathfrak{g}_{\mathbb{C}}$.

Definition 5.2.5. By a period map we mean a holomorphic locally liftable Griffiths transverse map

$$\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$$

for a smooth complex algebraic variety X and a finite index $\Gamma \subset \mathbf{G}(\mathbb{Z})$.

Remark 5.2.6. A period map $\varphi : X^{\mathrm{an}} \to \mathbf{G}(\mathbb{Z})\backslash D$ is equivalent to the data of a pure polarized integral variation of Hodge structures on X. This consists of:

- A local system $\mathscr{H}_{\mathbb{Z}}$ with a flat quadratic form $Q_{\mathbb{Z}}$.
- A holomorphic locally split filtration $F^\bullet$ of $\mathscr{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{X^{\mathrm{an}}}$ such that the flat connection ∇ satisfies Griffiths transversality:
$$\nabla(F^p) \subset F^{p-1} \text{ for all } p.$$
- We moreover require that $(\mathscr{H}_{\mathbb{Z}}, Q_{\mathbb{Z}}, F^\bullet)$ is fiberwise a pure polarized integral Hodge structure.

The period map lifts to $\Gamma\backslash D$ if Γ contains the image of the monodromy representation of $\mathscr{H}_{\mathbb{Z}}$.

Definition 5.2.7. Let $\overline{X}$ be a log smooth compactification of X. For any irreducible boundary divisor $E \subset \overline{X}$, the local monodromy operator $\gamma \in \mathbf{G}(\mathbb{Z})$ of E is the monodromy of the local system $\mathscr{H}_{\mathbb{Z}}$ along a small loop around E, which is defined up to conjugation (in $\mathbf{G}(\mathbb{Z})$).

The following result on the monodromy of variations of Hodge structures is of pervasive importance:

Theorem 5.2.8. *Any period map $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$ has quasiunipotent local monodromy.*

Corollary 5.2.9. *For any period map $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$, there is a finite étale cover $f : X' \to X$ such that the period map $\varphi' = \varphi \circ f : X' \to \Gamma\backslash D$ has unipotnent local monodromy.*

Proof. Note that any quasiunipotent $\gamma \in \mathbf{G}(\mathbb{Z})$ has eigenvalues which are roots of unity of bounded order. Let $\Gamma(n) \subset \mathbf{G}(\mathbb{Z})$ be the full-level n subgroup

$$\Gamma(n) := \left\{ \gamma \in \mathbf{G}(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\}.$$

Since the roots of unity of bounded order inject $\bmod\, p$ for suifficiently large p, it follows that every quasiunipotent element of $\Gamma(p)$ is in fact unipotent for sufficiently large p. Now take X' to be the pullback of the finite étale cover $\Gamma(p)\backslash D \to \mathbf{G}(\mathbb{Z})\backslash D$ (technically as stacks). □

5.3. The Mumford–Tate group and weakly special subvarieties.

Definition 5.3.1. Suppose $H_{\mathbb{Z}}$ carries a pure weight $2k$ Hodge structure. An integral (resp. rational) class $v \in H_{\mathbb{Z}}$ (resp. $v \in H_{\mathbb{Q}}$) is *Hodge* if $v \in H^{k,k}$.

Note that an integral class $v \in H_{\mathbb{Z}}$ has pure Hodge type if and only if it is a Hodge class. Moreover, v is Hodge if and only if $v \in F^k$.

Example 5.3.2. The motivation for considering Hodge classes again comes from geometry. Given a smooth projective complex algebraic variety Y and a closed algebraic subvariety $Z \subset Y$, the fundamental class $[Z] \in H^{2\,\mathrm{codim}\, Z}(Y, \mathbb{Z})$ is a Hodge class. The Hodge conjecture says that moreover all rational Hodge classes arise from cycles (up to rational scaling).

The Hodge classes of a particular Hodge structure are described by the Mumford–Tate group:

Definition 5.3.3. Suppose $H_{\mathbb{Q}}$ carries a pure Hodge structure H. The (special) Mumford–Tate group $\mathbf{MT}_H$ of H is the algebraic $\mathbb{Q}$-subgroup of $\mathbf{End}(H_{\mathbb{Q}})$ with the following property: for any tensor power $H' = H^{\otimes k} \otimes (H^\vee)^{\otimes \ell}$, the rational Hodge classes of H' are precisely the rational vectors fixed by $\mathbf{MT}_H$.

For simplicity we suppress the proof that such a group exists, as well as the relation to the Deligne torus, and we instead refer to [CMSP03] for details. Note that if the Hodge structure H is polarized by $q_{\mathbb{Q}}$, then $\mathbf{MT}_H \subset \mathbf{Aut}(H_{\mathbb{Q}}, q_{\mathbb{Q}})$.

Definition 5.3.4. Let D be a polarized period domain.

(1) A weak Mumford–Tate subdomain D' of D is an orbit $\mathbf{M}(\mathbb{R})x$ where $x \in D$ and $\mathbf{M}$ is a normal algebraic $\mathbb{Q}$-subgroup of $\mathbf{MT}_x$. In fact, D' is a smooth complex submanifold of D, and it is an irreducible component of the locus of Hodge structures H such that $\mathbf{MT}_H \supset \mathbf{M}$.
(2) If moreover $\mathbf{M} = \mathbf{MT}_x$, then $D' = \mathbf{M}(\mathbb{R})x$ is called a Mumford–Tate subdomain.
(3) Let $\pi : D \to \Gamma\backslash D$ be the quotient map. For $D' \subset D$ a (weak) Mumford–Tate subdomain, $\pi(D') \subset \Gamma\backslash D$ is a complex analytic subvariety which we call a (weak) Mumford–Tate subvariety. Likewise, given a period map $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$, we call $\varphi^{-1}\pi(D')$ a (weak) Mumford–Tate subvariety of X.

Given Definition 5.3.3, we see that we can also think of a Mumford–Tate subdomain as a component of the locus of Hodge structures for which some number of rational tensors are Hodge.

Theorem 5.3.5 (Theorem 1.6 of [CDK95])**.** *Let $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$ be a period map. Then any weak Mumford–Tate subvariety of X is algebraic.*

Remark 5.3.6. In the special case of $f : Y \to X$ a smooth projective family, and the period map corresponding to the variation of Hodge structures on $R^{2k}_{\mathrm{prim}} f_*\mathbb{Z}$, the Hodge conjecture implies Theorem 5.3.5. Indeed, the locus $\mathrm{Hdg}_k(X) \subset X$ where $H^{2k}_{\mathrm{prim}}(Y_x, \mathbb{Q})$ acquires Hodge classes is the image of the codimension k relative Hilbert scheme $\mathrm{Hilb}(Y/X)$, hence a countable union of algebraic subvarieties.

Definition 5.3.7. Suppose $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$ is a period map. The $\mathbb{Q}$-Zariski closure of the image of the monodromy representation $\varphi_* : \pi_1(X, x) \to \mathbf{G}(\mathbb{Q})$ is called the *algebraic monodromy group.*

The following theorem is a consequence of the theorem of the fixed part [CMSP03, Theorem 13.1.10], which asserts that the trivial sub-local system of a variation of Hodge structures naturally supports a Hodge sub-variation.

Theorem 5.3.8. *The identity component of the algebraic monodromy group of a period map is a $\mathbb{Q}$-factor of the very general Mumford–Tate group*[7]*.*

5.4. **Definable fundamental sets of period maps.**

We will need a slightly different definition of what a definable fundamental set of a period map is.

Definition 5.4.1. Let $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$ a period map. A definable fundamental set for φ is a definable space F whose underlying space is a complex analytic variety together with a commutative diagram of holomorphic maps

$$\begin{array}{ccc} F & \xrightarrow{\tilde{\varphi}} & D \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X^{\mathrm{an}} & \xrightarrow[\varphi]{} & \Gamma\backslash D \end{array}$$

such that p realizes X^{def} as a quotient by a closed étale definable equivalence relation and $\tilde{\varphi}$ is definable.

A crucial observation for the proof of the Ax–Schanuel conjecture is the following:

Proposition 5.4.2. *Any period map with unipotent monodromy admits a $\mathbb{R}_{\mathrm{an,exp}}$-definable fundamental set.*

The proof of Proposition 5.4.2 is not hard—it follows easily from the local description of degenerations of Hodge structures, as we will see below. For Proposition 5.4.2 the assumption on the monodromy is not necessary, but given Corollary 5.2.9 it is sufficient for our purposes to restrict to this case.

By a *local* period map we mean a holomorphic locally liftable Griffiths tranverse map

$$\varphi : (\Delta^*)^r \times \Delta^s \to \Gamma\backslash D$$

[7]That is, the Mumford–Tate group at a very general point

Given such a map, let $\mu : \mathbb{H}^r \times \Delta^s \to (\Delta^*)^r \times \Delta^s$ be the standard covering map, and consider a lift of the period map

$$\begin{array}{ccc} \mathbb{H}^r \times \Delta^s & \xrightarrow{\tilde{\varphi}} & D \\ \mu \downarrow & & \downarrow \pi \\ (\Delta^*)^r \times \Delta^s & \xrightarrow[\varphi]{} & \Gamma\backslash D \end{array}$$

The covering group of μ is $\mathbb{Z}^r$, generated by the real translations

$$t_i : \mathbb{H}^r \to \mathbb{H}^r : (z_1, \ldots, z_i, \ldots, z_r) \mapsto (z_1, \ldots, z_i + 1, \ldots, z_r)$$

on the ith $\mathbb{H}$ factor, for $1 \le i \le r$. Let $\gamma_i \in \mathbf{G}(\mathbb{Z})$ be the corresponding unipotent monodromy operator, so that

$$\tilde{\varphi} \circ (t_i \times \mathrm{id}_{\Delta^s}) = \gamma_i \tilde{\varphi}$$

Let

$$N_i := \log \gamma_i = -\sum_k \frac{(1-\gamma_i)^k}{k} \in \mathfrak{g}_{\mathbb{R}}$$

be the nilpotent logarithms of T_i, which makes sense since each T_i is unipotent. It follows that the map $\tilde{\psi} : \mathbb{H}^r \times \Delta^s \to D$ defined by "untwisting" the monodromy

$$\tilde{\psi} := \exp\left(-\sum_i z_i N_i\right) \tilde{\varphi}$$

descends to a map $\psi : (\Delta^*)^r \times \Delta^s \to D$.

Theorem 5.4.3 (Corollary 8.35 of [Sch73])**.** *For any local period map, ψ as defined above extends to a holomorphic map $\overline{\psi} : \Delta^n \to \check{D}$.*

Remark 5.4.4. Given a variation $(\mathscr{H}_{\mathbb{Z}}, Q_{\mathbb{Z}}, F^\bullet)$ of pure polarized integral Hodge structures over a smooth algebraic base X with unipotent local monodromy, the Deligne extension is a canonical extension of the associated flat bundle $\mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{Z}} \mathscr{H}_{\mathbb{Z}}$ to a log-smooth compactification $\overline{X}$ as a holomorphic vector bundle. For v_i a (multivalued) flat frame for $\mathscr{H}_{\mathbb{Z}}$ in a polydisk $(\Delta^*)^r \times \Delta^s$, the extension is defined using the frame

$$\tilde{v}_i := \exp\left(-\sum_i z_i N_i\right) v_i.$$

One then shows that these extensions patch to form a global extension of $\mathcal{O}_{X^{\mathrm{an}}} \otimes \mathscr{H}_{\mathbb{Z}}$ to $\overline{X}$ (see [BGK+87]). Theorem 5.4.3 then implies that the Hodge filtration $F^\bullet$ extends holomorphically to the Deligne extension.

Let $\Sigma \subset \mathbb{H}$ be the bounded vertical strip

$$\Sigma := \{z \in \mathbb{H} \mid -\epsilon < \mathrm{Re}\, z < 1 + \epsilon\}$$

with its $\mathbb{R}_{\mathrm{alg}}$-definable structure as a semialgebraic subset of $\mathbb{C}$. For $\delta > 0$ define

$$\begin{aligned} \Delta_\delta :=& \{q \in \Delta \mid |q| < 1 - \delta\} \\ \mathbb{H}_\delta :=& \{z \in \mathbb{H} \mid \mathrm{Im}\, z > \delta\} \\ \Sigma_\delta :=& \Sigma \cap \mathbb{H}_\delta. \end{aligned}$$

Corollary 5.4.5. *For all sufficiently small $\delta > 0$,*

$$\tilde{\varphi} : \Sigma_\delta^r \times \Delta_\delta^s \to D$$

is $\mathbb{R}_{\mathrm{an,exp}}$-definable.

Proof. We have $\varphi = \exp(z \cdot N)\tilde{\psi}$. By Theorem 5.4.3, $\psi : \Delta_\delta^n \to D$ is restricted-analytic, hence $\mathbb{R}_{\mathrm{an}}$-definable. It follows that $\tilde{\psi} : \Sigma_\delta^r \times \Delta_\delta^s \to D$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable since μ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. Now, $\mathbf{G}(\mathbb{C})$ (with its canonical definable structure) acts algebraically on $\check{D}$, and $\exp(z \cdot N)$ is in fact an algebraic map $\Sigma^r \to \mathbf{G}(\mathbb{C})$, hence $\mathbb{R}_{\mathrm{alg}}$-definable. Thus, $\tilde{\varphi}$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. □

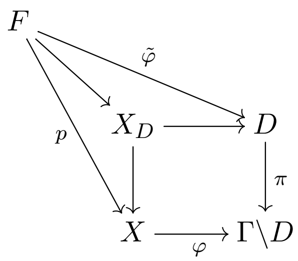
Proof of Proposition 5.4.2. Take an algebraic log smooth compactification $\overline{X}$ of X, and a finite cover of X by polydisks of the form $f_i : (\Delta^*)^{r_i} \times \Delta^{s_i} \to X$. For each such polydisk, take $F_i = \Sigma_\delta^{r_i} \times \Delta_\delta^{s_i}$, and let $p_i = f_i \circ \mu$. Finally, take $F = \bigsqcup_i F_i$, with $p = \bigsqcup_i p_i : F \to X$. For sufficiently small $\delta > 0$ the map

$$p : F \to X^{\mathrm{an}}$$

realizes X^{def} as a $\mathbb{R}_{\mathrm{an,exp}}$-definable quotient of F. By Corollary 5.4.5, the lifted period map $\tilde{\varphi} : F \to D$ is $\mathbb{R}_{\mathrm{an,exp}}$-definable. □

5.5. **Intersections with definable fundamental sets.**

Given a definable fundamental set for a period map as in the last subsection, we evidently have a natural diagram



where $X_D := X \times_{\Gamma\backslash D} D$. Fix a left-invariant metric h_D and let $\Phi = \tilde{\varphi}(F)$. For the proof in the next section, it will be important that a given algebraic subvariety $Z \subset \check{D}$ has bounded volume intersection with all translates of Φ under the action by $\mathbf{G}(\mathbb{Z})$. We in fact have the stronger statement:

Proposition 5.5.1 (Proposition 3.2 of [BT18])**.** *Let $Z \subset \check{D}$ be a closed algebraic subvariety. For all $\gamma \in \mathbf{G}(\mathbb{C})$,* $\mathrm{vol}(Z \cap \gamma\Phi) = O(1)$.

Proof. Evidently it is enough to show $\mathrm{vol}(Z' \cap \Phi) = O(1)$ for all Z' in the same connected component of the Hilbert scheme of $\check{D}$ as Z. Further, it suffices to show $\mathrm{vol}(\tilde{\varphi}^{-1}(Z')) = O(1)$ for each local period map $\tilde{\varphi} : \mathbb{H}_\delta^r \times \Delta_\delta^s \to D$ considered above, where the volume is computed with respect to $\tilde{\varphi}^* h_D$.

For any holomorphic Griffiths transverse map $f : M \to \Gamma\backslash D$ we have $f^* h_D \ll \kappa_M$ where κ_M is the Kobayashi metric of M. In particular, for $M = \mathbb{H}^r \times \Delta^s$ the metric κ_M is the maximum over the coordinate-wise Poincaré metrics. The factors in $\Sigma_\delta^r \times \Delta_\delta$ have finite volume with respect to the Kobayashi metric of $\mathbb{H}^r \times \Delta^s$, and thus it is enough to uniformly bound the degree of the projection of $\tilde{\varphi}^{-1}(Z')$ to any subset of coordinates. This in turn follows by applying Corollary 2.2.10 to the pullback of the universal family.

□

6. The Ax–Schanuel theorem for period maps

In this section we give the proof of the Ax–Schanuel conjecture for period maps from [BT18]. The proof follows the same strategy as the proof of Mok–Pila–Tsimerman [MPT17] for Shimura varieties.

6.1. Statement of the main theorem.

Let X be a smooth complex algebraic variety over $\mathbb{C}$ supporting a pure polarized integral variation of Hodge structures $\mathscr{H}_{\mathbb{Z}}$. Let $\mathbf{MT}_{\mathscr{H}_{\mathbb{Z}}}$ be the generic Mumford–Tate group—that is, the Mumford–Tate group at a very general point—and let $\Gamma \subset \mathbf{MT}_{\mathscr{H}_{\mathbb{Z}}}(\mathbb{Q})$ be the image of the monodromy representation $\pi_1(X) \to \mathbf{MT}_{\mathscr{H}_{\mathbb{Z}}}(\mathbb{Q})$ after possibly passing to a finite cover. Let $\mathbf{G}$ be the identity component of the $\mathbb{Q}$-Zariski closure of Γ. Let $D = D(\mathbf{G})$ be the associated weak Mumford–Tate domain and $\varphi : X \to \Gamma\backslash D$ the period map of $\mathscr{H}_{\mathbb{Z}}$. The compact dual $\check{D}$ of D is a projective variety containing D as an open set in the archimedean topology.

Consider the fiber product

$$\begin{array}{ccc} X \times D \supset X_D := X \times_{\Gamma\backslash D} D & \xrightarrow{\tilde{\varphi}} & D \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow[\varphi]{} & \Gamma\backslash D. \end{array}$$

Theorem 6.1.1 (Ax–Schanuel, Theorem 1.1 of [BT18])**.** *In the above setup, let $V \subset X \times \check{D}$ be an algebraic subvariety, and let U be an irreducible analytic component of $V \cap X_D$ such that*

$$\operatorname{codim}_{X\times D}(U) < \operatorname{codim}_{X\times\check{D}}(V) + \operatorname{codim}_{X\times D}(X_D).$$

Then the projection of U to X is contained in a proper weak Mumford–Tate subvariety.

The theorem for example implies that the (analytic) locus in X where the periods satisfy a given set of algebraic relations must be of the expected codimension unless there is a reduction in the generic Mumford–Tate group. See [Kliar] for some related discussions.

Corollary 6.1.2 (Ax–Lindemann–Weierstrass)**.** *Assume the above setup.*

(1) For any algebraic $V \subset D$, the Zariski closure of $\varphi^{-1}\pi(V)$ is a weak Mumford–Tate subvariety.
(2) For any algebraic $V \subset X$, the Zariski closure of any component V_0 of $\pi^{-1}\varphi(V)$ is a weak Mumford–Tate subdomain.

6.2. Setup for the proof.

Given a period map $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$ and a subvariety $V \subset X \times D$, we define its type as the tuple

$$(\dim X, \dim V - \dim(V \cap X_D), -\dim(V \cap X_D))$$

ordered lexicographically. We say a closed algebraic $V \subset X \times D$ is *bad* at $p \in V \cap X_D$ if

$$\operatorname{codim}_p(V \cap X_D) < \operatorname{codim}(V) + \operatorname{codim}(X_D)$$

in which case we also say that both p and V are *bad*.

We proceed by induction and assume the theorem for all smaller types. Suppose V_0 is bad with $N_0 = \dim(V_0 \cap X_D)$. Let $M \subset \operatorname{Hilb}(X \times \check{D})$ be the connected component of the Hilbert scheme containing V_0, let $\mathcal{V} \subset (X \times \check{D}) \times M$ be the universal subscheme, and let $\mathcal{V}_{X\times D} \subset (X \times D) \times M$ be the restriction of the universal family to $X \times D \subset X \times \check{D}$. We will refer to points of $\mathcal{V}_{X\times D}$ as pairs (p, V), with $V \in M$ and $p \in V \cap (X \times D)$.

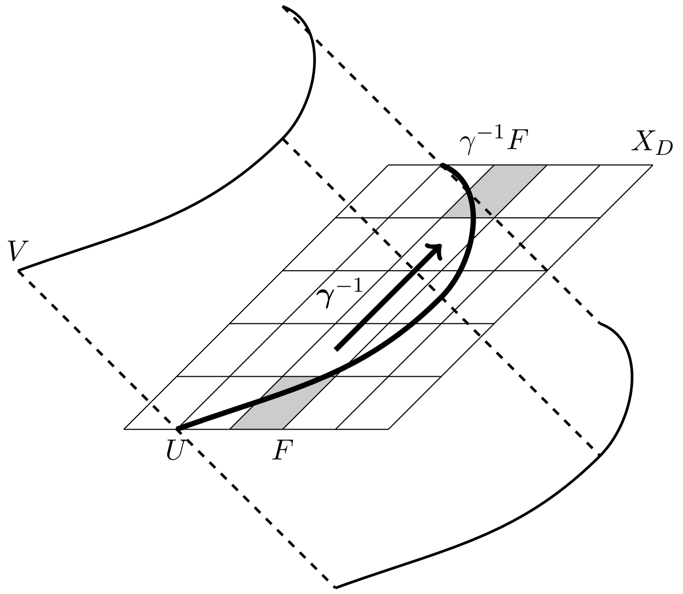


FIGURE 6. Every fundamental domain $\gamma^{-1}F$ that U passes through yields an integral translate γV that meets F badly.

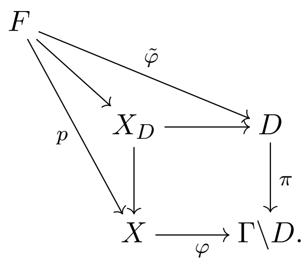
Let $\mathcal{V}_{X_D}$ be the universal intersection of $\mathcal{V}_{X\times D}$ with X_D. The set of "equally" bad points

$$\mathcal{B} := \{(p,V) \in \mathcal{V}_{X_D} \mid \dim_p(V \cap X_D) = N_0\} \subset \mathcal{V}_{X\times D}$$

is naturally a complex analytic subvariety which is moreover closed because of the inductive hypothesis (as $\dim_p(V \cap X_D)$ is semicontinuous). If $\mathcal{B} \to M$ is the projection $(p,V) \mapsto V$, the base-change $\mathcal{V}_{\mathcal{B}} \to \mathcal{B}$ of the universal family $\mathcal{V}_{X\times D}$ along $\mathcal{B} \to M$ is naturally the family of "equally" bad varieties V.

6.3. **Ingredients for the proof.**

Recall that we have a definable fundamental set in the sense of Definition 5.4.1:



Given a bad V, we would like to apply the Pila–Wilkie theorem to the set

$$I := \{g \in \mathbf{G}(\mathbb{R}) \mid \dim(gV \cap F) = N_0\} \tag{11}$$

of translates of V that meet F badly, just as in the proof of Theorem 4.1.3. Let Γ_X be the image of the monodromy representation $\pi_1(X) \to \mathbf{G}(\mathbb{Z})$. Once again, X_D is covered by fundamental sets $\gamma^{-1}F$ with $\gamma \in \mathbf{G}(\mathbb{Z})$, and if U is a N_0-dimensional component of $V \cap X_D$ then for each $\gamma^{-1}F$ that U meets we certainly have $\gamma \in I$ (see Figure 6). We would like to argue that U passes through many fundamental sets, and therefore I has many integral points.

Like in the Shimura variety case, however, the monodromy is now very complicated and we cannot make the "height balls" argument work, so we instead use metric balls. We may assume U meets F and take a basepoint $x_0 \in F \cap U$. Let y_0 be the image in $\check{D}$, and consider the radius r ball $B_{p_0}(r)$ centered at y_0 with respect to the natural left-invariant metric on D. In the following we always measure volumes of subsets of $X \times D$ with respect to a left-invariant volume form on the second factor.

For $\gamma \in \Gamma_X$ we have $V \cap \gamma^{-1}F = U \cap \gamma^{-1}F$, as the component of X_D containing U is fixed by Γ_X. Now, by Proposition 5.5.1, U meets each $\gamma^{-1}F$ with bounded volume, while the $\gamma^{-1}F$ meet each other with bounded multiplicity, and it follows that the number of $\gamma^{-1}F$ that U passes through in $X \times B_{p_0}(r)$ is at least as much (up to a constant) as its volume in $X \times B_{y_0}(r)$. Given the following theorem, this volume grows exponentially in r:

Theorem 6.3.1 (Theorem 1.2 of [BT18])**.** *There are constants $\beta, R > 0$ such that for any closed positive-dimensional Griffiths-transverse anaytic subvariety $Z \subset B_{y_0}(r)$ for $r > R$ we have*

$$\mathrm{vol}(Z) \gg e^{\beta r}\, \mathrm{mult}_{y_0} Z.$$

On the other hand, the fundamental sets $\gamma^{-1}F$ which intersect $X \times B_{y_0}(r)$ have height which is at most exponential in the radius:

Theorem 6.3.2 (Theorem 4.2 of [BT18])**.** *For any $\gamma \in \mathbf{G}(\mathbb{Z})$ with*

$$\gamma^{-1}F \cap (X \times B_{y_0}(r)) \neq \varnothing$$

we have $H(\gamma) = e^{O(r)}$.

Putting Theorems 6.3.1 and 6.3.2 together we therefore obtain:

Proposition 6.3.3. *For some $\epsilon > 0$,*

$$N(I, t) \gg t^{\epsilon}.$$

We postpone a precise definition of the height function on $\mathbf{G}(\mathbb{Q})$ and $N(I, t)$ until the next lecture. In the remainder of this section, we prove Theorem 6.1.1 assuming Proposition 6.3.3, and discuss the proofs of Theorems 6.3.2 and 6.3.1 in Lectures 7 and 8, respectively.

6.4. **The counting step.**

We can now adapt the argument of Lecture 4 to first show:

Proposition 6.4.1. $\mathrm{Stab}_{\mathbf{G}(\mathbb{Z})}(V)$ *is infinite for any fiber V of $\mathcal{V}_{\mathcal{B}}$.*

Proof. $\mathbf{G}$ is an algebraic group, so $\mathbf{G}(\mathbb{R})$ has a natural definable structure. Exactly as in the proof of Theorem 4.1.3, the set I from (11) is $\mathbb{R}_{\mathrm{an,exp}}$-definable, and therefore by Proposition 6.3.3 and the Pila–Wilkie theorem we conclude that I contains a semialgebraic curve $C \subset I$ containing arbitrarily many integer points, in particular at least 2 integer points.

If cV is constant in $c \in C$, then it follows that V is stabilized by a non-identity integer point and we are done (since Γ is torsion free). So we assume that cV varies with $c \in C$. Note that since C contains an integer point that $\tilde{\varphi}(cV \cap X_D)$ is not contained in a weak Mumford-Tate subdomain for at least one $c \in C$, and thus for all but a countable subset of C (since there are only countably many families of weak Mumford–Tate subdomains).

We now have two cases to consider (see Figure 7). On the one hand, assume there is no fixed N_0-dimensional component U of $cV \cap X_D$ as $c \in C$ varies. Then we may replace V by $\bigcup_{c \in C} cV$ and increase both $\dim V$ and $\dim(V \cap X_D)$ by one, thus lowering the type and contradicting the inductive hypothesis. On the

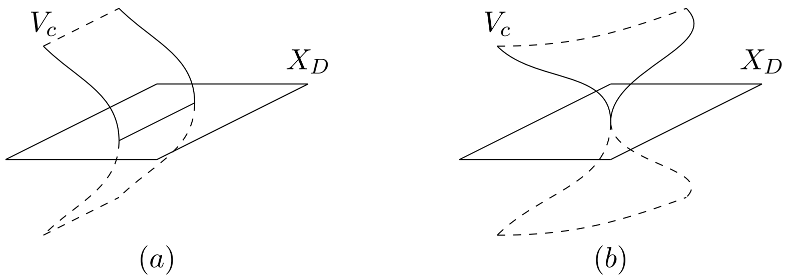


FIGURE 7. If V is not stabilized by $c \in C$, then we get a counterexample with smaller type by replacing V with either (a) $\bigcup_c cV$ or (b) $\bigcap_c cV$.

other hand, if there is such a component, then replacing V with $\bigcap_{c\in C} cV$ we lower $\dim V$ without changing $\dim(V \cap X_D)$, again contradicting the inductive assumption. This completes the proof.

□

6.5. **The definable Chow step.**

Now we would like to control how many bad points of X there are. Obviously such points are Zariski dense, for replacing X by the Zariski closure we contradict the inductive assumption on $\dim X$. However, the bad points may *a priori* be quite sparse.

Proposition 6.5.1. *The projection $\mathcal{B} \to X$ is surjective.*

Proof. The universal intersection $\mathcal{V}_{X_D}$ is proper over X_D, and the restriction $\mathcal{V}_F \subset F \times M$ has a canonical definable structure as a restriction of an algebraic subvariety of $(X \times \check{D}) \times M$ to $F \times M$. The quotient $\mathcal{V}_X := \Gamma_X \backslash \mathcal{V}_{X_D}$ is a complex analytic space, proper over X, which thereby inherits a unique definable structure for which the quotient map $\mathcal{V}_F \to \mathcal{V}_X$ is definable.

Likewise, the subset

$$\mathcal{B}_F := \{(p, V) \in \mathcal{V}_F \mid \dim_p(V \cap F) = N_0\}$$

is a definable closed complex analytic subset of $\mathcal{V}_F$, and the quotient is a definable closed complex analytic subset $\mathcal{B}_X \subset \mathcal{V}_X$ which is proper over X.

To finish, the projection $\mathcal{B}_X \to X$ is a proper definable complex analytic map, and by Remmert–Stein and Proposition 2.1.2 the image $Z \subset X$ is a definable closed complex analytic subvariety of X, and therefore algebraic by Theorem 3.2.1. We must then have $Z = X$, by the induction hypothesis. □

Corollary 6.5.2. *The image of $\pi_1(\mathcal{B}_X) \to \pi_1(X)$ is finite-index.*

Proof. $\mathcal{B}_X \to X$ is a proper surjective map of complex analytic varieties. □

6.6. **The induction step.**

In the final step we produce a contradiction to Proposition 6.4.1.

Proposition 6.6.1. $\mathrm{Stab}_{\mathbf{G}(\mathbb{Z})}(V)$ *is finite for a very general fiber V of $\mathcal{V}_{\mathcal{B}}$.*

The crucial point is that Hodge theory relates the monodromy of a variation of Hodge structures to the Mumford–Tate group of a very general fiber. Theorem 5.3.8 will therefore imply a reduction in the Mumford–Tate group which cannot occur by the inductive hypothesis.

Proof. By the construction in the previous step we have $\mathcal{B}_X = \Gamma_X\backslash\mathcal{B}$, and the fundamental group $\pi_1(\mathcal{B}_X)$ naturally acts on $\mathcal{B}$. Explicitly, if ρ is the composition $\pi_1(\mathcal{B}_X) \to \pi_1(X) \to \mathbf{G}(\mathbb{Z})$, then for $\gamma \in \pi_1(\mathcal{B}_X)$ this action is $(p, V) \mapsto (\rho(\gamma)p, \rho(\gamma)V)$. Let $\Gamma_{\mathcal{B}}$ be the image of ρ, and note that by Corollary 6.5.2 that $\Gamma_{\mathcal{B}}$ is $\mathbb{Q}$-Zariski dense in $\mathbf{G}$.

As $\mathbf{G}(\mathbb{R})$ acts on $X \times \check{D}$ by algebraic automorphisms, given $g \in \mathbf{G}(\mathbb{R})$ the locus in M of varieties V stabilized by g is an algebraic subvariety of M. It follows that for the fibers of the family $\mathcal{V}_{\mathcal{B}} \to \mathcal{B}$ outside of a countable collection of proper subvarieties of $\mathcal{B}$—that is, for the very general fiber V—the stabilizer under $\mathbf{G}(\mathbb{Z})$ is a fixed group $\Gamma_{\mathcal{V}}$. Furthermore, for a very general fiber V, γV is also very general for any $\gamma \in \Gamma_{\mathcal{A}}$, and it follows that $\Gamma_{\mathcal{V}}$ is normalized by $\Gamma_{\mathcal{B}}$. Letting $\mathbf{\Theta}$ be the identity component of the $\mathbb{Q}$-Zariski closure of $\Gamma_{\mathcal{V}}$, we conclude that $\mathbf{\Theta}$ is a normal $\mathbb{Q}$-subgroup of $\mathbf{G}$.

It suffices to show that the $\mathbb{Q}$-Zariski closure of $\Gamma_{\mathcal{V}}$ is finite, or that:

Claim. $\mathbf{\Theta}$ is the identity subgroup.

Proof. Since $\mathbf{\Theta}$ is a normal $\mathbb{Q}$-subgroup by construction, $\mathbf{G}$ is isogenous to $\mathbf{\Theta}_1 \times \mathbf{\Theta}_2$ with $\mathbf{\Theta}_2 = \mathbf{\Theta}$. We have a splitting of weak Mumford–Tate domains $D = D_1 \times D_2$ with $D_i = D(\mathbf{\Theta}_i)$. Replacing X by a finite cover we also have a splitting of the period map [GGK12, Theorem III.A.1]

$$\varphi = \varphi_1 \times \varphi_2 : X \to \Gamma_1\backslash D_1 \times \Gamma_2\backslash D_2.$$

Moreover, φ_1, φ_2 satisfy Griffiths transversality (see the proof of [GGK12, Theorem III.A.1]). Note that $V \subset X \times D$ by assumption, and as V is invariant under $\mathbf{\Theta}_2$ it is of the form $V_1 \times D_2$ where $V_1 \subset X \times D_1$.

Consider the period map $X \to \Gamma_1\backslash D_1$, the resulting $X_{D_1} \subset X \times D_1$, and the subvariety $V_1 \subset X \times D_1$. Let U be a N_0-dimensional component of $V \cap X_D$ and let U_1 be the component of $V_1 \cap X_{D_1}$ onto which U projects. By assumption the theorem applies in this situation, and as U_1 cannot be contained in a proper weak Mumford–Tate subdomain (for then U would as well), we must have

$$\operatorname{codim}_{X\times D_1}(U_1) = \operatorname{codim}_{X\times\check{D}_1}(V_1) + \operatorname{codim}_{X\times D_1}(X_{D_1}).$$

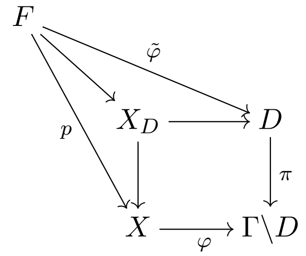
Note that the projection $X_D \to X_{D_1}$ has discrete fibers, so $\dim X = \dim X_{D_1}$ and $\dim U = \dim U_1$, whereas $\operatorname{codim} V_1 = \operatorname{codim} V$, which is a contradiction if φ_2 is non-constant.

□

□

7. Heights and distances

In this section we establish the comparison between heights and metric dilation needed in Theorem 6.3.2. Recall that we have a period map $\varphi : X^{\mathrm{an}} \to \Gamma\backslash D$ and definable fundamental set $p : F \to X$ in the sense of Definition 5.4.1 consisting of a union of unwrapped polydisks[8] $\Sigma^r \times \Delta^s$:



Letting $\Phi = \tilde{\varphi}(F)$ and fixing a basepoint $x_0 \in \Phi \subset D$, we identify $D \cong \mathbf{G}(\mathbb{R})/V$ for a compact subgroup $V \subset \mathbf{G}(\mathbb{R})$. Thinking of D as a space of Hodge structures on the fixed integral lattice $(H_{\mathbb{Z}}, q_{\mathbb{Z}})$, as before we denote by h_x the induced Hodge metric on $H_{\mathbb{C}}$ corresponding to $x \in D$.

Definition 7.0.1. For $\gamma \in \mathbf{G}(\mathbb{Z})$ let $H(\gamma)$ be the height of γ with respect to the representation $\rho_{\mathbb{Z}} : \mathbf{G}(\mathbb{Z}) \to \mathrm{GL}(H_{\mathbb{Z}})$. For $g \in \mathbf{G}(\mathbb{R})$, we denote by $||\rho_{\mathbb{R}}(g)||$ the maximum archimedean size of the entries of $\rho_{\mathbb{R}}(g)$, so that if $\gamma \in \mathbf{G}(\mathbb{Z})$ we have $H(\gamma) = ||\rho_{\mathbb{R}}(\gamma)||$.

Remark 7.0.2. We can now precisely define the counting function used in the previous lecture. For $U \subset \mathbf{G}(\mathbb{R})$ a definable subset (where $\mathbf{G}(\mathbb{R})$ is given the canonical definable structure coming from the algebraic group structure), we define

$$N(U, t) := \#\{\gamma \in U \cap \mathbf{G}(\mathbb{Q}) \mid H(\gamma) \leq t\}.$$

By fixing a V-invariant hermitian metric at x_0, we obtain a left-invariant hermitian metric on $\mathbf{G}(\mathbb{R})/V$. This metric is explicitly described as follows. For any point $x \in D$ we have seen in Remark 5.2.4 that the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ inherits a polarized Hodge structure $\mathfrak{g}_x$, and that the tangent space T_xD is identified with $T_xD = \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}}$. The space $\mathfrak{g}^{<0} := \bigoplus_{p<0} \mathfrak{g}^{p,-p} \subset \mathfrak{g}_{\mathbb{C}}$ provides a real analytic lift of T_xD, and we endow T_xD with the restriction of the hodge metric h_x on $\mathfrak{g}_x$. One can easily check that this metric is left-invariant.

For any $R > 0$ let $B_{x_0}(R) \subset D$ be the ball of radius R centered at x_0. The main goal of this section is to establish the following:

Theorem 7.0.3 (Theorem 4.2 of [BT18])**.** *Any $\gamma \in \mathbf{G}(\mathbb{Z})$ with*

$$\gamma^{-1}\Phi \cap B_{x_0}(r) \neq \varnothing$$

has $H(\gamma) = e^{O(r)}$.

Define $d_0(x) = d(x, x_0)$. We write $f \preceq g$ if $|f| \ll |g|^{O(1)} + O(1)$, and $f \asymp g$ if $f \preceq g$ and $g \preceq f$.

Lemma 7.0.4. Let $\lambda(x, x')$ be the maximal eigenvalue of h_x with respect to $h_{x'}$. Then

(1) For all $g \in \mathbf{G}(\mathbb{R})$ we have $||\rho_{\mathbb{R}}(g)|| \asymp e^{d_0(gx_0)}$;
(2) $\lambda(x, x') \asymp e^{d(x,x')}$.

Proof. Let $K = U(h_{x_0}) \cap \mathbf{G}(\mathbb{R})$ be the subgroup of $\mathbf{G}(\mathbb{R})$ acting unitarily with respect to h_{x_0}. Then K is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ containing V, and the above left-invariant metric on $G(\mathbb{R})/V$ descends to the symmetric

[8]Recall from the discussion following Theorem 5.4.3 that $\Sigma \subset \mathbb{H}$ is a bounded vertical strip

$$\Sigma := \{z \in \mathbb{H} \mid -\epsilon < \operatorname{Re} z < 1 + \epsilon\}.$$

space $\mathbf{G}(\mathbb{R})/K$. Note that the diameters of the fibers of $\mathbf{G}(\mathbb{R})/V \to \mathbf{G}(\mathbb{R})/K$ are bounded. Choosing a K-orthogonal split maximal torus $A \subset \mathbf{G}(\mathbb{R})$ and a basis A_i of the Lie algebra $\mathfrak{a}$ of A, the induced metric on A is up to scaling the unique left-invariant metric, which is identified with the euclidean metric on the Lie algebra $\mathfrak{a}$. We therefore have for any $g \in \mathbf{G}(\mathbb{R})$ with KAK decomposition $g = k_1 a k_2$

$$\sqrt{\sum_i t_i^2} \ll d_0(gx_0) = d_0(ax_0) + O(1) \ll \sqrt{\sum_i t_i^2} + O(1)$$

where $a = \exp(\sum_i t_i A_i)$. As

$$\max_i \exp(|t_i|) \preceq \rho_{\mathbb{R}}(g) \preceq \max_i \exp(|t_i|)$$

part (1) follows.

For part (2), note that by $\mathbf{G}(\mathbb{R})$-invariance we may restrict to the case $x' = x_0$. Setting $\rho = \rho_{\mathbb{R}}$ for convenience, note that $\operatorname{tr}(\rho(g)^*\rho(g))$ is a sum of the eigenvalues of h_{gx_0} with respect to h_{x_0}, where $\rho(g)^*$ is the adjoint of $\rho(g)$ with respect to h_{x_0}. Thus $\operatorname{tr}(\rho(g)^*\rho(g)) \asymp \lambda(gx_0, x_0)$. As $\operatorname{tr}(\rho(g)^*\rho(g))$ is the sum of the squares of the entries of $\rho(g)$, part (2) follows from part (1).

□

We define a function $\mu : D \to \mathbb{R}$ measuring proximity to the boundary by the minimal period length:

$$\mu(x) = \min_{v \in H_{\mathbb{Z}} \backslash \{0\}} h_x(v).$$

For any $v \in H_{\mathbb{C}}$ we have $\log \frac{h_{x_0}(v)}{h_x(v)} \ll d_0(x) + O(1)$ by part (2) of Lemma 7.0.4, and so we deduce the following:

Corollary 7.0.5. $-\log \mu \ll d_0 + O(1)$.

Proof. There is some $v \in H_{\mathbb{Z}} \backslash \{0\}$ with $\log \mu(x) = \log h_x(v)$ and thus

$$-\log \mu = -\log h_x(v) \ll \log \frac{h_{x_0}(v)}{h_x(v)} + O(1) \ll d_0(x) + O(1)$$

where we have used that h_{x_0} is comparable to a standard Hermitian metric on $H_{\mathbb{C}}$, so that $h_{x_0}(v) \gg 1$ for any $v \in H_{\mathbb{Z}} \backslash \{0\}$. □

We in fact have a comparison in the other direction once we restrict to Φ:

Lemma 7.0.6. For $x \in \Phi$ we have $d_0(x) \ll -\log \mu(x) + O(1)$.

The proof uses the asymptotics of hodge norms, which we now recall. Given a local period map $\varphi : (\Delta^*)^r \times \Delta^s \to \Gamma \backslash D$ with unipotent local monodromy let $\tilde{\varphi} : \Sigma^r \times \Delta^s \to D$ be a lift and $N_1, \ldots, N_r$ the nilpotent monodromy logarithms. The monodromy logarithms (with the implicit chosen ordering of the coordinates of $(\Delta^*)^r$) define r weight filtrations $W^{(j)} = W(N_1, \ldots, N_j)$. For a given $v \in H_{\mathbb{Z}}$, let $w^{(j)}$ be its weight with respect to $W^{(j)}$—that is, for each j, we take $w^{(j)}$ to be the unique w such that $v \in W^{(j)}_w$ and $\operatorname{gr}^{W^{(j)}}_w(v) \neq 0$. By [CKS86], on the region

$$\operatorname{Im} z_1 \gg \cdots \operatorname{Im} z_r \gg 1$$

the hodge norm $h_{\varphi(z)}(v)$ of v at $\varphi(z)$ is then given asymptotically by

$$h_{\tilde{\varphi}(z)}(v) \sim \left(\frac{\operatorname{Im} z_1}{\operatorname{Im} z_2}\right)^{w^{(1)}} \cdots \left(\frac{\operatorname{Im} z_{r-1}}{\operatorname{Im} z_r}\right)^{w^{(r-1)}} \cdot (\operatorname{Im} z_r)^{w^{(r)}}$$

where "$\sim$" means "within a bounded function of."

Proof of Lemma 7.0.6. It is enough to prove the statement for the image of a single $\Sigma^r \times \Delta^s$. Moreover, we may cover F with finitely many regions S_π of the form $\operatorname{Im} z_{\pi(1)} \gg \cdots \operatorname{Im} z_{\pi(\ell)} \gg 1$ where π ranges over all permutations of $\{1, \ldots, r\}$. Thus, we may assume Φ is the image of S_{id}.

Take v_i to be a basis of $H_{\mathbb{Z}}$ descending to a basis of the multi-graded module associated to the r weight filtrations $W^{(j)}$ as above, where we take each grading centered at 0. Let $w_i^{(j)}$ for $j = 1, \ldots, r$ be the weights of v_i with respect to $W(\pi)^{(j)}$. As above, on S_{id} we therefore have

$$h_{\tilde{\varphi}(z)}(v_i) \sim \left(\frac{\operatorname{Im} z_1}{\operatorname{Im} z_2}\right)^{w_i^{(1)}} \cdots \quad \left(\frac{\operatorname{Im} z_{r-1}}{\operatorname{Im} z_r}\right)^{w_i^{r-1}} \cdot \quad (\operatorname{Im} z_r)^{w_i^{(r)}}.$$

As the set of weights is preserved under negation, it follows that $\max_i h_{\tilde{\varphi}(z)}(v_i) \sim (\min_i h_{\tilde{\varphi}(z)}(v_i))^{-1}$, and so by Lemma 7.0.4,

$$d_0(\tilde{\varphi}(z)) \ll \max_i \log h_{\tilde{\varphi}(z)}(v_i) \ll -\log \mu(\tilde{\varphi}(z)) + O(1)$$

uniformly on every such region. □

Proof of Theorem 6.3.2. Suppose $x \in B_0(R) \cap \gamma^{-1}\Phi$ for $\gamma \in \mathbf{G}(\mathbb{Z})$. Putting together Lemma 7.0.6 and Corollary 7.0.5 we have

$$d_0(\gamma x) \ll -\log \mu(\gamma x) + O(1) = -\log \mu(x) + O(1) \ll d_0(x) + O(1)$$

and since

$$d_0(\gamma x_0) \leq d(\gamma x, \gamma x_0) + d(\gamma x, x_0) \leq d_0(x) + d_0(\gamma x)$$

we are finished by part (1) of Lemma 7.0.4. □

8. Volume bounds

In this lecture we outline the proof of the volume bound in Theorem 6.3.1. To warm up for the proof, we first give a simple proof in the euclidean case.

8.1. **Euclidean space.** Endow $\mathbb{C}^n$ with the standard hermitian metric

$$h_{\mathrm{eucl}} = \sum_i dz_i \otimes d\bar{z}_i.$$

The real part

$$\operatorname{Re} h_{\mathrm{eucl}} = \sum_i dx_i^2 + dy_i^2$$

is the usual euclidean metric on $\mathbb{C}^n = \mathbb{R}^{2n}$, and the associated Kähler form is

$$\omega_{\mathrm{eucl}} := -\operatorname{Im} h_{\mathrm{eucl}} = \frac{1}{2}\sum_i i dz_i \wedge d\bar{z}_i = \sum_i dx_i \wedge dy_i.$$

Given a locally closed analytic subvariety $Z \subset \mathbb{C}^n$, its euclidean volume can be computed as

$$\mathrm{vol}_{\mathrm{eucl}}(Z) := \frac{1}{(\dim Z)!}\int_Z (\omega_{\mathrm{eucl}})^{\dim Z}.$$

Finally, for $z_0 \in \mathbb{C}^n$ denote by

$$B_{z_0}^{\mathrm{eucl}}(R) := \{z \in \mathbb{C}^n \mid |z - z_0|^2 < R\}$$

the radius R ball around z_0 with respect to h_{eucl}.

Theorem 8.1.1. *For any $z_0 \in \mathbb{C}^n$ and any closed analytic subvariety $Z \subset B_{z_0}^{\mathrm{eucl}}(R) \subset \mathbb{C}^n$, we have*

$$\mathrm{vol}(Z) \geq (\pi R^2)^{\dim Z} \cdot \mathrm{mult}_{z_0} Z.$$

The theorem is originally due to Federer (see for example [Sto66]). Note moreover that the bound is sharp, as a union of N affine linear spaces through z_0 will realize the bound. Hwang–To [HT02] have generalized the theorem to bounded symmetric domains, and it is their approach that we follow—and in fact that will generalize to the period domain setting.

The proof hinges on two observations: on the one hand, the "distance to z_0" function $\nu_{z_0}(z) := |z - z_0|^2$ provides a potential for ω_{eucl},

$$\omega_{\mathrm{eucl}} = \frac{i}{2}\partial\overline{\partial}\nu_{z_0}$$

while on the other hand, the log-distance $\log \nu_{z_0}$ is the potential for a form (strictly speaking, a current) that computes the multiplicity, by the Poincaré–Lelong formula.

Proof of Theorem 8.1.1. We may as well assume $z_0 = 0$ and set $\nu := \nu_{z_0}$. Set $Z(r) := Z \cap B_0^{\mathrm{eucl}}(r)$. By Stokes' theorem we have

$$\begin{aligned}
\mathrm{vol}^{\mathrm{eucl}}(Z(r)) &= \int_{Z(r)} (\tfrac{i}{2}\partial\overline{\partial}\nu)^{\dim Z} \\
&= \int_{\partial Z(r)} \tfrac{1}{2} d^c\nu \wedge (\tfrac{i}{2}\partial\overline{\partial}\nu)^{\dim Z - 1} \\
&= r^2 \cdot \int_{\partial Z(r)} \tfrac{1}{2} d^c \log\nu \wedge (\tfrac{i}{2}\partial\overline{\partial}\nu)^{\dim Z - 1} \\
&= r^2 \cdot \int_{Z(r)} \tfrac{i}{2}\partial\overline{\partial}\log\nu \wedge (\tfrac{i}{2}\partial\overline{\partial}\nu)^{\dim Z - 1}.
\end{aligned}$$

Note that in going from the second to the third line, we used that ν is constant on $\partial Z(r)$, so for any function $f : \mathbb{R} \to \mathbb{R}$ we have

$$d^c f(\nu)|_{\partial Z(r)} = (f'(\nu) d^c\nu)|_{\partial Z(r)} = f'(r) \cdot d^c\nu|_{\partial Z(r)}.$$

Carrying out the same manipulation for each $\frac{i}{2}\partial\overline{\partial}\nu$ term we arrive at

$$\mathrm{vol}^{\mathrm{eucl}}(Z(r)) = r^{2\dim Z} \cdot \int_{Z(r)} (\tfrac{i}{2}\partial\overline{\partial}\log\nu)^{\dim Z}. \tag{12}$$

Without getting into the details (see for example [HT02]), we briefly remark that some care must be taken in the above wedge product as $\partial\overline{\partial}\log\nu$ must be interpreted as a current in order for Stoke's theorem to apply.

For the remaining part of the argument, let's for simplicity assume Z is a curve, so that we have a normalization of the form $g : \Delta \to Z(\epsilon)$ (with $g(0) = 0$), for some sufficiently small $\epsilon > 0$. Now, as $\log\nu$ is plurisubharmonic we have

$$\begin{aligned}
\int_{Z(r)} \tfrac{i}{2}\partial\overline{\partial}\log\nu &\geq \int_{Z(\epsilon)} \tfrac{i}{2}\partial\overline{\partial}\log\nu \\
&= \int_{\Delta} \tfrac{i}{2}\partial\overline{\partial}\log g^*\nu \\
&= \int_{\Delta} \tfrac{i}{2}\partial\overline{\partial}\log |t|^{2\,\mathrm{mult}_0 Z} \\
&= \frac{i}{2}\int_{S^1} \frac{d\overline{t}}{\overline{t}} \cdot \mathrm{mult}_0 Z \\
&= \pi \cdot \mathrm{mult}_0 Z.
\end{aligned}$$

□

From the proof, we can conclude the following statement about the growth of the volume:

Proposition 8.1.2. *In the situation of Theorem 8.1.1,*

$$\frac{\mathrm{vol}^{\mathrm{eucl}}(Z \cap B^{\mathrm{eucl}}_{z_0}(r))}{r^{2\dim Z}}$$

is a non-decreasing function of r *for* $0 < r < R$.

Proof. Immediate from (12), as $\log \nu$ is plurisubharmonic and thus

$$\int_{Z(r)} (\tfrac{i}{2}\partial\overline{\partial}\log\nu)^{\dim Z}$$

is a nondecreasing function of r. □

Remark 8.1.3. Let's say a few words about the last step of the above proof for those who are unfamiliar with multiplicity in the analytic category. Suppose z_i are the standard coordinates of $\mathbb{C}^n$, and suppose $g : \Delta \to Z(\epsilon)$ is the normalization considered in the proof. Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the local ring of germs of analytic functions at 0, $m_{\mathbb{C}^n,0} \subset \mathcal{O}_{\mathbb{C}^n,0}$ the ideal of the origin, and $I_{Z,0} \subset \mathcal{O}_{\mathbb{C}^n,0}$ the ideal of Z. We have

$$\begin{aligned} \mathrm{mult}_0 Z &:= \max\{k \in \mathbb{N} \mid m^k_{\mathbb{C}^n,0} \supset I_{Z,0}\} \\ &= \min_i \mathrm{ord}_0\, g^* z_i. \end{aligned}$$

8.2. **Period domains.**

Let D be a polarized period domain equipped with its natural left-invariant hermitian metric and associated positive $(1,1)$ form ω. We would now like to adapt the ideas from the previous subsection to prove:

Theorem 8.2.1. *There are constants* $\beta, \rho > 0$ *(only depending on* D*) such that for any* $R > \rho$*, any* $x_0 \in D$*, and any positive-dimensional Griffiths transverse closed analytic subvariety* $Z \subset B_{x_0}(R) \subset D$*, we have*

$$\mathrm{vol}(Z) \geq e^{\beta R}\, \mathrm{mult}_{x_0} Z$$

where $B_{x_0}(R)$ *is the radius* R *ball centered at* x_0 *and* $\mathrm{vol}(Z)$ *the volume with respect to the natural left-invariant metric on* D.

The crux of the proof is to find an exhaustion function $\varphi_0 : D \to \mathbb{R}$ which on the one hand defines balls

$$B^{\varphi_0}(R) : \{X \in D \mid \varphi_0(x) < R\}$$

that are comparable to the metric balls $B_{x_0}(R)$ and on the other hand is a potential for a $(1,1)$ form that is comparable to ω in the Griffiths transverse directions. The difficulty is that unlike in the euclidean case (or indeed even the bounded symmetric domain case) Theorem 8.2.1 fails without the Griffiths-tranverse assumption, as D contains compact subvarieties in the vertical directions. Thus, the function φ_0 must necessarily treat the Griffiths transverse directions in a special way.

We state the precise properties of the function φ_0 in the following proposition, but first introduce some notation.

Definition 8.2.2.

(1) Given a real $(1,1)$ form α on D, we say $\alpha \geq_{\mathrm{trans}} 0$ if at point $x \in D$ and any Griffiths transverse $X \in T^{1,0}_x D$ we have

$$-i\alpha_x(X, \overline{X}) \geq 0.$$

(2) Given two real $(1,1)$ forms α, β on D, we say that $\alpha = O_{\mathrm{trans}}(\beta)$ if for some positive constant $C > 0$ we have
$$C\beta - \alpha \geq_{\mathrm{trans}} 0.$$

Now fix $x_0 \in D$ and denote by $d_0 : D \to \mathbb{R}$ the distance function to x_0.

Proposition 8.2.3. *There is a smooth function $\varphi_0 : D \to \mathbb{R}$ with the following properties:*

(1) $d_0(x) \ll \varphi_0(x) + O(1)$ and $\varphi_0(x) \ll d_0(x) + O(1)$;
(2) $i\partial\overline{\partial}\varphi_0 \geq_{\mathrm{trans}} 0$ and $i\partial\overline{\partial}\varphi_0 >_{\mathrm{trans}} 0$ at x_0;
(3) $i\partial\overline{\partial}\varphi_0 = O_{\mathrm{trans}}(\omega)$ and $|\partial\varphi_0|^2 = O_{\mathrm{trans}}(i\partial\overline{\partial}\varphi_0)$.

Proof. See [BT18]. □

Assuming Proposition 8.2.3, we can now complete the proof of Theorem 6.3.1. For any closed Griffiths transverse analytic subvariety $Z \subset B(R) \subset D$ of dimension d, define
$$\mathrm{vol}^{\varphi_0}(Z) := \frac{1}{d!}\int_Z (i\partial\overline{\partial}\varphi_0)^d.$$
We begin with the following:

Proposition 8.2.4. *There is a constant $\beta > 0$ such that for any $R > 0$ and any positive-dimensional Griffiths transverse closed analytic subvariety $Z \subset B^{\varphi_0}(R)$,*
$$e^{-\beta r}\,\mathrm{vol}^{\varphi_0}(Z \cap B^{\varphi_0}(r))$$
is a nondecreasing function in $r \in [0, R]$.

Proof. Let $d = \dim Z$. Let $\psi_0 = -e^{-\beta\varphi_0}$ for $\beta > 0$ the constant such that
$$i\partial\overline{\partial}\varphi_0 - \beta|\partial\varphi_0|^2 \geq_{\mathrm{trans}} 0$$
which is guaranteed by Proposition 8.2.33. We then have
$$i\partial\overline{\partial}\psi_0 = \beta e^{-\beta\varphi_0}\left(i\partial\overline{\partial}\varphi_0 - \beta|\partial\varphi_0|^2\right) \geq_{\mathrm{trans}} 0.$$
By Stokes' theorem we have
$$\begin{aligned}
\mathrm{vol}^{\varphi}(Z \cap B^{\varphi_0}(r)) &= \int_{Z\cap B^{\varphi_0}(r)} (i\partial\overline{\partial}\varphi_0)^d \\
&= \int_{Z\cap\partial B^{\varphi_0}(r)} d^c\varphi_0 \wedge (i\partial\overline{\partial}\varphi_0)^{d-1} \\
&= \beta^{-1}e^{\beta r}\int_{Z\cap\partial B^{\varphi_0}(r)} d^c\psi_0 \wedge (i\partial\overline{\partial}\varphi_0)^{d-1} \\
&= \beta^{-1}e^{\beta r}\int_{Z\cap B^{\varphi_0}(r)} i\partial\overline{\partial}\psi_0 \wedge (i\partial\overline{\partial}\varphi_0)^{d-1} \\
&= \beta^{-d}e^{\beta d r}\int_{Z\cap B^{\varphi_0}(r)} (i\partial\overline{\partial}\psi_0)^d
\end{aligned}$$
which implies the claim, as $\psi_0|_Z$ is plurisubharmonic. □

Proof of Theorem 8.2.1. Choose a fixed euclidean ball B centered around x_0 with respect to some coordinate system. By Theorem 8.1.1 we have an inequality of the form
$$\mathrm{vol}^{\mathrm{eucl}}(Z \cap B) \gg \mathrm{mult}_{x_0} Z$$
Choose a fixed radius ρ such that $B \subset B^{\varphi_0}(\rho)$. After possibly shrinking B, $i\partial\overline{\partial}\varphi_0$ is comparable to the euclidean Kähler form on B in Griffiths transverse directions

by Proposition 8.2.3(2), and combining this with the previous proposition we have

$$\mathrm{vol}^{\varphi_0}(Z \cap B^{\varphi_0}(r)) \gg e^{\beta r}\,\mathrm{vol}^{\varphi_0}(Z \cap B^{\varphi_0}(\rho)) \gg e^{\beta r}\,\mathrm{mult}_{x_0} Z$$

for all $r > \rho$.

Now, by Proposition 8.2.3(1), after possibly increasing ρ, there is a constant $C > 0$ such that

$$B_{x_0}(r) \supset B^{\varphi_0}(Cr)$$

for all $r > \rho$, so

$$\mathrm{vol}^{\varphi_0}(Z \cap B_{x_0}(r)) \gg e^{\beta r}\,\mathrm{mult}_{x_0} Z$$

for all $r > \rho$. Finally, by Proposition 8.2.3(3) we have

$$\mathrm{vol}(Z \cap B_{x_0}(r)) \gg \mathrm{vol}^{\varphi_0}(Z \cap B_{x_0}(r))$$

and the claim follows. □

Remark 8.2.5. Theorem 8.2.1 has a number of interesting applications in its own right. They lie outside the scope of these notes, but we briefly describe one to give a flavor. We say a point $x \in \Gamma\backslash D$ has injectivity radius R if the ball $B_x(R) \subset D$ injects into $\Gamma\backslash D$. For a period map $\varphi : X \to \Gamma\backslash D$, Theorem 8.2.1 then says that the Seshadri constant of the Hodge bundle at a point $x \in X$ can be bounded by the injectivity radius of $\varphi(x)$. In particular, these Seshadri constants can be made to grow in the level covers of X. See [HT06] for some related applications in the context of Shimura varieties using the volume bounds of Hwang–To.

9. Further directions

9.1. **Derivatives.**

One can generalize the transcendence statements by considering not only automorphic funtions, but also their derivatives. For example, in the case of the modular curve one has the parametrization $j : \mathbb{H} \to Y(1)$,and j satisfies a 3rd degree differential equation. In this context, building on work of Pila [Pil13], the paper [MPT17] proves the following generalization of the modular Ax-Schanuel statement:

Theorem 9.1.1. *Let $z_1, \ldots, z_n$ be meromorphic germs in auxiliary variables t_i at some point of $\mathbb{H}^n$, and assume that none of the z_i are constant, nor are $\mathrm{SL}_2(\mathbb{Q})$ translates of each other. Then*

$$\mathrm{trdeg}_{\mathbb{C}}\,\mathbb{C}\left(z_1, j(z_1), j'(z_1), j''(z_1), \ldots, z_n, j(z_n), j'(z_n), j''(z_n)\right) \geq 3n + \mathrm{rk}\left(\frac{\partial z_j}{\partial t_i}\right).$$

Note that the above is much stronger then the usual Ax-Schanuel as it includes the algebraic independence of the derivatives of j as well. One may also generalize (as [MPT17] does) to arbitrary Shimura varieties, but in that generality one cannot easily pick out distinguished variables. Therefore the paper adopts the language of jet spaces to formulate the above statement. The proofs are much the same, except one has to keep track of jet spaces in all the geometric constructions.

9.2. **Definability of period maps.**

In Lecture 4 we showed that weakly special subvarieties of Shimura varieties were algebraic in two steps: first by using the existence of a definable fundamental set to argue that weakly special subvarieties are definable complex analytic subvarieties and second by appealing to the definable Chow theorem.

To use the same argument to reprove Theorem 5.3.5, we must have two ingredients:

(1) $\mathbf{G}(\mathbb{Z})\backslash D$ must be given a S-definable structure for some o-minimal S, and (weak) Mumford–Tate subvarieties must be shown to be definable with respect to this structure.
(2) Period maps $\varphi : X^{\mathrm{an}} \to \mathbf{G}(\mathbb{Z})\backslash D$ from a complex algebraic variety X must be shown to be definable with respect to this definable structure.

Accomplishing (1) and (2) is the content of [BKT18]. For (1), we define an arithmetic quotient (of a homogeneous space) to be

$$\Gamma\backslash\mathbf{G}(\mathbb{R})/V$$

for $\mathbf{G}$ a connected semi-simple algebraic $\mathbb{Q}$-group, $\Gamma \subset \mathbf{G}(\mathbb{Q})$ an arithmetic lattice, $V \subset \mathbf{G}(\mathbb{R})$ a connected compact subgroup. We moreover define a morphism

$$\Gamma\backslash\mathbf{G}(\mathbb{R})/V \to \Gamma'\backslash\mathbf{G}'(\mathbb{R})/V'$$

of arithmetic quotients to be a map arising from a morphism $f : \mathbf{G} \to \mathbf{G}'$ of algebraic $\mathbb{Q}$-groups sending Γ to Γ' and V to V'.

Theorem 9.2.1 (Theorem 1.1 of [BKT18])**.** *Every arithemtic quotient has a natural $\mathbb{R}_{\mathrm{alg}}$-definable structure with respect to which every morphism of arithmetic quotients is $\mathbb{R}_{\mathrm{alg}}$-definable.*

Briefly, the definable structure is built by using a Siegel set to construct a definable fundamental set. Theorem 9.2.1 is easily seen to imply the required statement about weak Mumford-Tate subvarieties of arithmetic quotients of period domains.

Theorem 9.2.2 (Theorem 1.3 of [BKT18])**.** *Let X be a smooth complex algebraic variety. Any period map*

$$\varphi : X^{\mathrm{an}} \to \mathbf{G}(\mathbb{Z})\backslash D$$

is $\mathbb{R}_{\mathrm{an,exp}}$-definable with respect to the $\mathbb{R}_{\mathrm{an,exp}}$-definable structure[9] *on $\mathbf{G}(\mathbb{Z})\backslash D$ induced from Theorem 9.2.1.*

The crux of the proof of Theorem 9.2.2 is to show that lifts of local period maps (as in 5.4) land in *finitely* many Siegel sets. In addition to the norm asymptotics discussed in Lecture 7, the primary ingredient is the SL_2-orbit theorem of Schmid [Sch73].

Corollary 9.2.3 (Theorem 1.6 of [BKT18])**.** *Every weak Mumford–Tate subvariety of X is algebraic.*

9.3. Definable GAGA.

Let S be an o-minimal structure. There is a natural notion of S-definable complex analytic varieties—loosely speaking, they are complex analytic varieties with a finite holomorphic atlas by S-definable complex analytic subvarieties of $\mathbb{C}^n$ with S-definable holomorphic transition functions. As first examples we have $\mathbb{G}_m^{\mathrm{def}}$ and $\mathbb{C}_a^*$ for each $a \in \mathbb{R}$ from Example 2.3.2. Some care is needed to define the sheaf of S-definable holomorphic functions, as it will only satisfy the sheaf axiom with respect to S-definable—in particular finite—covers. Thus, it is naturally a sheaf on the *S-definable site* of the underlying S-definable space. The category of definable complex analytic varieties is introduced in [BBT18].

[9]and the canonical $\mathbb{R}_{\mathrm{an,exp}}$-definable structure on X.

Let (AlgSp/$\mathbb{C}$) be the category of separated algebraic spaces[10] that are finite type over $\mathbb{C}$, (An/$\mathbb{C}$) the category of complex analytic spaces, and (S-An/$\mathbb{C}$) the category of S-definable complex analytic spaces. The definabilization functor of Lecture 2 can be upgraded to a functor

$$(\mathrm{AlgSp}/\mathbb{C}) \to (S\text{-}\mathrm{An}/\mathbb{C}) : X \mapsto X^{\mathrm{def}}$$

which fits into a diagram

$$\begin{array}{ccc} (\mathrm{AlgSp}/\mathbb{C}) & \xrightarrow{(-)^{\mathrm{def}}} & (S\text{-}\mathrm{An}/\mathbb{C}) \\ & {\scriptstyle (-)^{\mathrm{an}}} \searrow \quad \swarrow {\scriptstyle (-)^{\mathrm{an}}} & \\ & (\mathrm{An}/\mathbb{C}) & \end{array}$$

where $(\mathrm{AlgSp}/\mathbb{C}) \to (\mathrm{An}/\mathbb{C}) : X \mapsto X^{\mathrm{an}}$ is now the usual analytification functor. Moreover, there is a natural definabilization functor on coherent sheaves

$$(-)^{\mathrm{def}} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{def}}).$$

Recall that GAGA says that for X a proper separated algebraic space of finite type over $\mathbb{C}$, the analytification functor on coherent sheaves

$$(-)^{\mathrm{an}} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{an}})$$

is an equivalence of categories. As a companion to the definable Chow theorem in Lecture 3, we have the following definable GAGA:

Theorem 9.3.1 (Theorem 1.3 of [BBT18])**.** *Let S be an o-minimal structure. Let X be a separated algebraic space of finite type over $\mathbb{C}$ and X^{def} the associated definable analytic space. The definabilization functor $(-)^{\mathrm{def}} : \mathbf{Coh}(X) \to \mathbf{Coh}(X^{\mathrm{def}})$ is fully faithful, exact, and its essential image is closed under subobjects and quotients.*

Thus in particular definable coherent subsheaves of algebraic coherent sheaves are algebraic.

Note that $(-)^{\mathrm{def}}$ is *not* essentially surjective in general. The reason for this is as follows. By definable cell decomposition, it is not hard to see that there is a definable cover of X^{eucl} by simply-connected (definable) subspaces. It follows that any $\mathbb{C}$-local system L is definable, and therefore that the coherent sheaf $F := L \otimes_{\mathbb{C}_X} \mathcal{O}_{X^{\mathrm{def}}}$ is definable, but analytic sections with the prescribed monodromy may easily fail to be definable. See [BBT18, Example 3.2] for details.

9.4. **Definable images.**

By combining the definable GAGA theorem with algebraization theorems of Artin, it is proven in [BBT18] that proper definable images of algebraic varieties are algebraic:

Theorem 9.4.1 (Theorem 1.4 of [BBT18])**.** *Let S be an o-minimal structure. Let X be a separated algebraic space of finite type over $\mathbb{C}$, $\mathcal{S}$ a definable analytic space, and $\varphi : X^{\mathrm{def}} \to \mathcal{S}$ a proper definable analytic map. Then $\varphi : X^{\mathrm{def}} \to \varphi(X^{\mathrm{def}})$ is (uniquely) the definabilization of a map of algebraic spaces.*

This can be used to resolve a conjecture of Griffiths [Gri70, pg.259] on the quasiprojectivity of images of period maps. For a pure polarized integral variation of Hodge structures $(\mathscr{H}_{\mathbb{Z}}, F^\bullet, q_{\mathbb{Z}})$, we define the Griffiths bundle to be $L := \bigotimes^i \det F^i$.

[10]One could consider the category of schemes that are of finite type over $\mathbb{C}$ for simplicity.

Theorem 9.4.2 (Theorem 1.1 of [BBT18])**.** *Let X be a reduced separated algebraic space of finite type over $\mathbb{C}$ and $\varphi : X^{\mathrm{an}} \to \Gamma\backslash\Omega$ a period map as above. Then*

(1) φ factors (uniquely) as $\varphi = \iota \circ f^{\mathrm{an}}$ where $f : X \to Y$ is a dominant map of (reduced) finite-type algebraic spaces and $\iota : Y^{\mathrm{an}} \to \Gamma\backslash\Omega$ is a closed immersion of analytic spaces;

(2) the Griffiths $\mathbb{Q}$-bundle L restricted to Y is the analytification of an ample algebraic $\mathbb{Q}$-bundle, and in particular Y is a quasi-projective variety.

Theorem 9.4.2 in turn has a number of applications; we refer to [BBT18] for related discussions.

References

[And92] Yves André. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compositio Math.*, 82(1):1–24, 1992.

[Ax71] James Ax. On Schanuel's conjectures. *Ann. of Math. (2)*, 93:252–268, 1971.

[Ax72] James Ax. Some topics in differential algebraic geometry. I. Analytic subgroups of algebraic groups. *Amer. J. Math.*, 94:1195–1204, 1972.

[BB66] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[BBT18] B. Bakker, Y. Brunebarbe, and J. Tsimerman. o-minimal GAGA and a conjecture of Griffiths. `arXiv:1811.12230`, 2018.

[BGK+87] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers. *Algebraic D-modules*, volume 2 of *Perspectives in Mathematics*. Academic Press, Inc., Boston, MA, 1987.

[Bis64] Errett Bishop. Conditions for the analyticity of certain sets. *Michigan Math. J.*, 11:289–304, 1964.

[BKT18] B. Bakker, B. Klingler, and J. Tsimerman. Tame topology of arithmetic quotients and algebraicity of Hodge loci. `arXiv:1810.04801`, 2018.

[BMZ99] E. Bombieri, D. Masser, and U. Zannier. Intersecting a curve with algebraic subgroups of multiplicative groups. *Internat. Math. Res. Notices*, (20):1119–1140, 1999.

[BMZ08] E. Bombieri, D. Masser, and U. Zannier. On unlikely intersections of complex varieties with tori. *Acta Arith.*, 133(4):309–323, 2008.

[BP89] E. Bombieri and J. Pila. The number of integral points on arcs and ovals. *Duke Math. J.*, 59(2):337–357, 1989.

[BT18] B. Bakker and J. Tsimerman. The Ax-Schanuel conjecture for variations of hodge structures. `arXiv:1712.05088`, 2018.

[CDK95] E. Cattani, P. Deligne, and A. Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.

[CKS86] E. Cattani, A. Kaplan, and W. Schmid. Degeneration of Hodge structures. *Ann. of Math. (2)*, 123(3):457–535, 1986.

[CMSP03] J. Carlson, S. Müller-Stach, and C. Peters. *Period mappings and period domains*, volume 85 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.

[Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Fal84] G. Faltings. Die Vermutungen von Tate und Mordell. *Jahresber. Deutsch. Math.-Verein.*, 86(1):1–13, 1984.

[Gao17] Ziyang Gao. Towards the Andre–Oort conjecture for mixed Shimura varieties: The Ax–Lindemann theorem and lower bounds for Galois orbits of special points. *J. Reine Angew. Math.*, 732:85–146, 2017.

[Gao18] Z. Gao. Mixed Ax-Schanuel for the universal abelian varieties and some applications. `arXiv:1806.01408`, 2018.

[GGK12] M. Green, P. Griffiths, and M. Kerr. *Mumford-Tate groups and domains*, volume 183 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012.

[Gri70] Phillip A. Griffiths. Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems. *Bull. Amer. Math. Soc.*, 76:228–296, 1970.

[Hab09] P. Habegger. On the bounded height conjecture. *Int. Math. Res. Not. IMRN*, (5):860–886, 2009.

[HT02] J. Hwang and W. To. Volumes of complex analytic subvarieties of Hermitian symmetric spaces. *American Journal of Mathematics*, 124(6):1221–1246, 2002.

[HT06] J.-M. Hwang and W.-K. To. Uniform boundedness of level structures on abelian varieties over complex function fields. *Mathematische Annalen*, 335(2):363–377, 2006.

[Kliar] B. Klingler. Hodge loci and atypical intersections: conjectures. In *Motives and Complex Multiplication, (Monte-Verita, 2016).* to appear.

[KUY16] B. Klingler, E. Ullmo, and A. Yafaev. The hyperbolic Ax-Lindemann-Weierstrass conjecture. *Publ. Math. Inst. Hautes Études Sci.*, 123:333–360, 2016.

[KY14] Bruno Klingler and Andrei Yafaev. The André-Oort conjecture. *Ann. of Math. (2)*, 180(3):867–925, 2014.

[Lan02] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics.* third edition, 2002.

[LV18] B. Lawrence and A. Venkatesh. Diophantine problems and p-adic period mappings. `arXiv:1807.02721`, 2018.

[Mau08] Guillaume Maurin. Courbes algébriques et équations multiplicatives. *Math. Ann.*, 341(4):789–824, 2008.

[Mil13] J. S. Milne. Shimura varieties and moduli. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 467–548. Int. Press, Somerville, MA, 2013.

[Moo98] Ben Moonen. Linearity properties of Shimura varieties. II. *Compositio Math.*, 114(1):3–35, 1998.

[MPT17] N. Mok, J. Pila, and J. Tsimerman. Ax-Schanuel for Shimura varieties. `arXiv:1711.02189`, 2017.

[Pil11] J. Pila. O-minimality and the André-Oort conjecture for $\mathbb{C}^n$. *Ann. of Math. (2)*, 173(3):1779–1840, 2011.

[Pil13] Jonathan Pila. Modular Ax-Lindemann-Weierstrass with derivatives. *Notre Dame J. Form. Log.*, 54(3-4):553–565, 2013.

[Pil14a] Jonathan Pila. O-minimality and Diophantine geometry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. 1*, pages 547–572. Kyung Moon Sa, Seoul, 2014.

[Pil14b] Jonathan Pila. O-minimality and Diophantine geometry. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. 1*, pages 547–572. Kyung Moon Sa, Seoul, 2014.

[Pin05] Richard Pink. A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang. Preprint, available from the authors website, 2005.

[PS03] Y. Peterzil and S. Starchenko. Expansions of algebraically closed fields. II. Functions of several variables. *J. Math. Log.*, 3(1):1–35, 2003.

[PT14] J. Pila and J. Tsimerman. Ax-Lindemann for $\mathscr{A}_g$. *Ann. of Math. (2)*, 179(2):659–681, 2014.

[PT16] J. Pila and J. Tsimerman. Ax-Schanuel for the j-function. *Duke Math. J.*, 165(13):2587–2605, 2016.

[PW06] J. Pila and A. J. Wilkie. The rational points of a definable set. *Duke Math. J.*, 133(3):591–616, 2006.

[Ray83a] M. Raynaud. Courbes sur une variété abélienne et points de torsion. *Invent. Math.*, 71(1):207–233, 1983.

[Ray83b] M. Raynaud. Sous-variétés d'une variété abélienne et points de torsion. In *Arithmetic and geometry, Vol. I*, volume 35 of *Progr. Math.*, pages 327–352. Birkhäuser Boston, Boston, MA, 1983.

[RSW03] J.-P. Rolin, P. Speissegger, and A. J. Wilkie. Quasianalytic Denjoy-Carleman classes and o-minimality. *J. Amer. Math. Soc.*, 16(4):751–777, 2003.

[Sch73] W. Schmid. Variation of Hodge structure: the singularities of the period mapping. *Invent. Math.*, 22:211–319, 1973.

[Sei58] A. Seidenberg. Abstract differential algebra and the analytic case. *Proc. Amer. Math. Soc.*, 9:159–164, 1958.

[Sto66] G. Stolzenberg. *Volumes, limits, and extensions of analytic varieties.* Lecture Notes in Mathematics, No. 19. Springer-Verlag, Berlin-New York, 1966.

[Tsi18] J. Tsimerman. The André-Oort conjecture for A_g. *Ann. of Math. (2)*, 187(2):379–390, 2018.

[UY11] E. Ullmo and A. Yafaev. A characterization of special subvarieties. *Mathematika*, 57(2):263–273, 2011.

[vdD98] Lou van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[vdDM96] Lou van den Dries and Chris Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 84(2):497–540, 1996.

[Wil99] A. J. Wilkie. A theorem of the complement and some new o-minimal structures. *Selecta Math. (N.S.)*, 5(4):397–421, 1999.

[Zil02] Boris Zilber. Exponential sums equations and the Schanuel conjecture. *J. London Math. Soc. (2)*, 65(1):27–44, 2002.

B. Bakker: Dept. of Mathematics, University of Georgia, Athens, USA.
E-mail address: bakker@math.uga.edu

J. Tsimerman: Dept. of Mathematics, University of Toronto, Toronto, Canada.
E-mail address: jacobt@math.toronto.edu