Baily–Borel compactifications and *b*-semiampleness

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The classical story: $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$

(1) (Satake '56) $A_g^{\mathrm{SBB}} = A_g \sqcup A_{g-1} \sqcup \cdots \sqcup A_1 \sqcup A_0$.

(2) (Baily '58) $A_g^{\mathrm{SBB}} =$ Proj(graded ring of automorphic forms).

(3) modular interpretation.

$$\begin{tikzcd}
X^\circ \arrow[r, hook] \arrow[d] & X \arrow[d] & X_0 \arrow[l, hook'] \arrow[d] \\
C^\circ \arrow[r, hook] & C \arrow[r, phantom, "\ni"] & 0
\end{tikzcd}$$

- X°/C° a family of abelian varieties
- X/C is the (semistable) Neron model
- Identity comp. of X_0 is a semi-abel. variety
- Compact part is limiting point in boundary

(4) natural polarization. $\pi : X \to \mathcal{A}_g$ the universal family, $L_{\mathcal{A}_g} = \det \pi_* \Omega_{X/\mathcal{A}_g}$ extends to an ample bundle $L_{A_g^{\mathrm{SBB}}}$ (up to a power).

The classical story: $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$

(5) universality. (Borel '72)

$$\begin{tikzcd}
Y^\circ := Y\backslash D \arrow[r, hook] \arrow[d, "f"'] & Y \arrow[d, dashed, "\overline{f}"] \\
\mathcal{A}_g \arrow[r] & A_g^{\mathrm{SBB}}
\end{tikzcd}$$

- (Y, D) log smooth
- $L_{Y^\circ} := f^* L_{\mathcal{A}_g}$ has an extension L_Y
- $\overline{f}^* L_{A_g^{\mathrm{SBB}}} \cong L_Y$

(5') Even have (5) in the analytic category, i.e. $(Y, D) = (\Delta^k, \text{coordinate hyperplanes})$

(Satake '60, Baily–Borel '66) Generalized to arbitrary arithmetic locally symmetric varieties.

Period maps

Let (Y, D) log smooth proper

$\pi : X^\circ \to Y^\circ$ be a smooth projective family,

$V = R^k \pi_* \mathbb{Z}_{(X^\circ)^{\mathrm{an}}}$ equipped with its polarizable $\mathbb{Z}$-VHS (filtration $F^\bullet V$ on $\mathcal{O}_{(Y^\circ)^{\mathrm{an}}} \otimes_{\mathbb{Z}_{(Y^\circ)^{\mathrm{an}}}} V$).

$$\begin{tikzcd}
(Y^\circ)^{\mathrm{an}} \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & \Gamma\backslash\mathbb{D} \\
& \mathcal{Z} := \overline{\operatorname{img}\varphi} \arrow[ur, hook] &
\end{tikzcd}$$

Questions (Griffiths '70).

(A) Is $\mathcal{Z}$ algebraic?

(B) Is Griffiths bundle $L_{\mathcal{Z}} := \bigotimes_p \det F^p V$ algebraic? Ample?

(C) Is there a $\mathcal{Z}^{\mathrm{BB}}$?

Main theorem 1

Question (Griffiths '70).

(A) Is $\mathcal{Z}$ algebraic?

(B) Is $L_{\mathcal{Z}} := \bigotimes_p \det F^p V$ algebraic? Ample?

Theorem (B–Brunebarbe–Tsimerman '23)

$\left((Y^\circ)^{\mathrm{an}} \xrightarrow{\psi} \mathcal{Z}\right) = \left(Y^\circ \xrightarrow{f} Z\right)^{\mathrm{an}}$ and $L_{\mathcal{Z}} = (L_Z)^{\mathrm{an}}$ all algebraic, L_Z ample.

(C) Is there a $\mathcal{Z}^{\mathrm{BB}}$?

Theorem 1 (B–Filipazzi–Mauri–Tsimerman)

- $B_Z := \bigoplus_k H^0_{mg}(Z, L_Z^k)$ *is finitely generated.*
- $Z^{\mathrm{BB}} :=$ Proj B_Z *is projective compactification of* Z *to which* L_Z *extends amply and universally, as in (5) (even (5')).*

Theorem 1 (B–Filipazzi–Mauri–Tsimerman)

Let Z be the image of any period map.

- $B_Z := \bigoplus_k H^0_{mg}(Z, L_Z^k)$ *is finitely generated.*
- $Z^{\mathrm{BB}} := \operatorname{Proj} B_Z$ *is projective compactification of Z to which L_Z extends amply and universally, as in (5) (even (5')).*

Corollary

Any moduli space with a local Torelli theorem has a canonical minimal Hodge–theoretic compactification.

- Z^{BB} is stratified by subvarieties with quasifinite period maps—the ones coming from the associated graded of the limit mixed Hodge structures.
- Lots of previous work of Green–Griffiths–Laza–Robles and Green–Griffiths–Robles, including some special cases. Green–Griffiths–Robles establish key ingredient of our proof.

Moduli of Calabi–Yau varieties

For $\pi : X^\circ \to Y^\circ$ a family of Calabi–Yau m-folds, $V = R^m\pi_*\mathbb{Z}_{(X^\circ)^{\mathrm{an}}}$. What about the *Hodge bundle* $M_{Y^\circ} = \pi_*\omega_{X^\circ/Y^\circ} = F^mV$?

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

Let Z be a coarse moduli space of smooth polarized CY varieties.

- $C_Z := \bigoplus_k H^0_{mg}(Z, M_Z^k)$ *is finitely generated.*
- $Z^{\mathrm{BBH}} := \operatorname{Proj} C_Z$ *is a projective compactification of Z to which* **the Hodge bundle** M_Z *extends amply and universally, as in (5/5').*

- Also works for klt Calabi–Yau pairs.
- Implies the b-semiampleness conjecture of Mori, Kawamata, Shokurov, Ambro, and Prokhorov–Shokurov. Partial past results of Ambro, Lazić, Floris,...
- There is a morphism $Z^{\mathrm{BB}} \to Z^{\mathrm{BBH}}$.
- Z^{BBH} has a stratification parametrizing minimal lc centers of degenerations (themselves CY!), up to finite ambiguity.

Semipositive line bundles in Hodge theory

For (Y, D) proper log smooth, let $(V, F^\bullet V)$ on Y° be a pol. $\mathbb{Z}$-VHS.

- Theorem 1 implies $L_Y = \bigotimes_p \det F^p V$ is semiample.
- Theorem 2 implies $M_Y = F^{\mathrm{top}} V$ is semiample if V is the middle cohomology of a family of CYs.
- In fact each $\det F^p V$ is semipositive...but NOT always semiample.

There is a unifying framework:

If the deepest part of $F^\bullet V$ is a line bundle, we call V a CY $\mathbb{Z}$-VHS, and call $M_{Y^\circ} = F^{\mathrm{top}} V$ the *Hodge bundle*.

Example. $\det F^p V = \mathrm{Hodge}\left(\bigwedge^{\operatorname{rk} F^p V} V\right)$.

Example. $\mathrm{Griffiths}\,(V) = \mathrm{Hodge}\left(\bigotimes_p \bigwedge^{\operatorname{rk} F^p V} V\right)$.

When is a Hodge bundle semiample?

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

Let (Y, D) be a proper log smooth algebraic space and $(V, F^\bullet V)$ a pol. CY $\mathbb{Z}$-VHS on Y°. Then M_Y is semiample if and only if it is **integrable** *and* **has torsion combinatorial monodromy**.

Integrability. If M_Y is flat on some analytic germ T, it is flat on the Zariski closure T^{Zar}.

Torsion combinatorial monodromy. If M_Y is numerically trivial on a connected curve, it is torsion.

Ok what about these two conditions?

For the Griffiths bundle of any VHS...

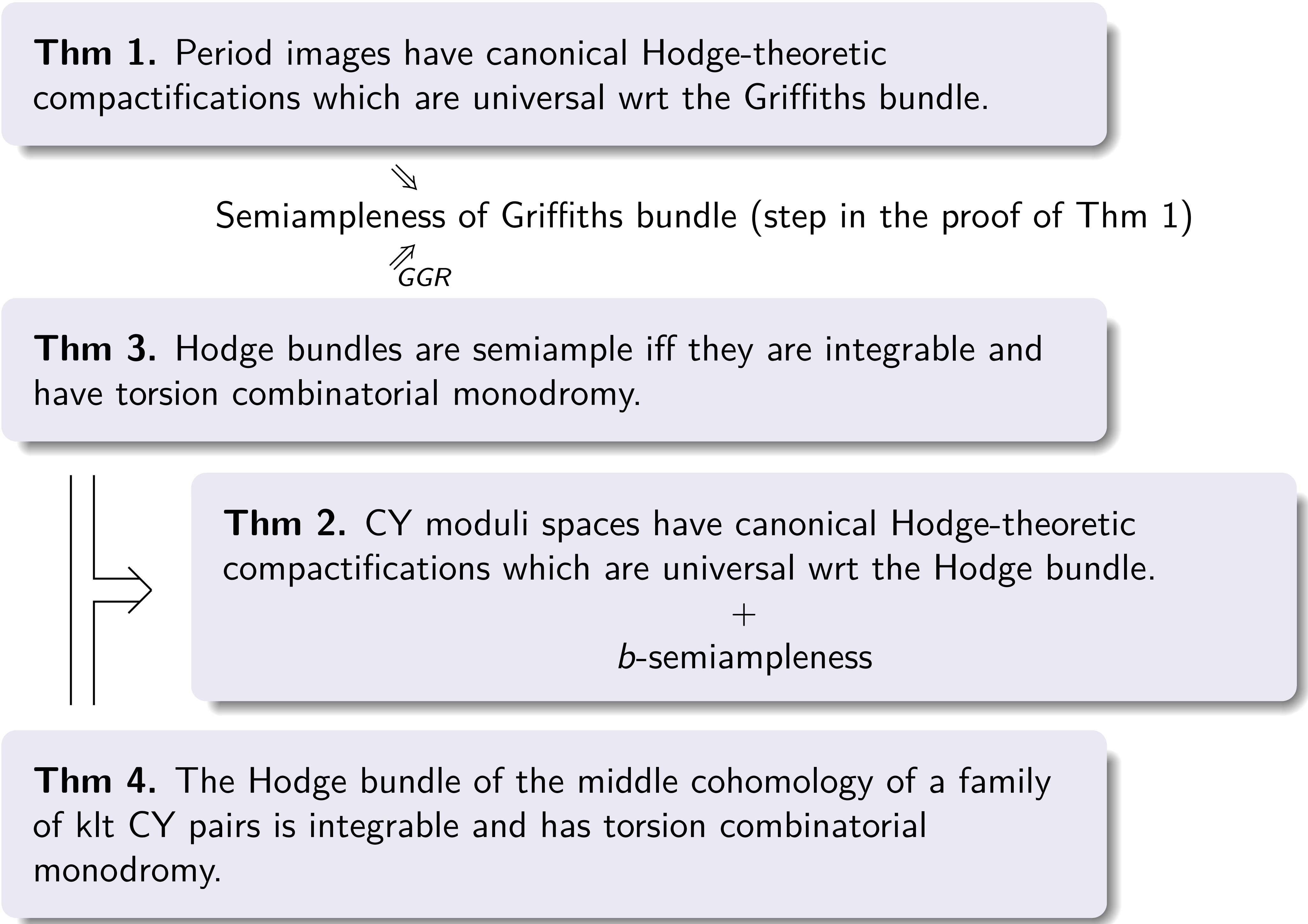
- Integrability is *automatic*.
- **Thm(Green–Griffiths–Robles).** The Griffiths bundle has torsion combinatorial monodromy.

For the Hodge bundle of a family of CY varieties...

Theorem 4 (B–Filipazzi–Mauri–Tsimerman)

(Y, D) proper log smooth, $(V, F^\bullet V)$ the pol. CY $\mathbb{Z}$-VHS on Y° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

Key. Even in the boundary, the Hodge bundle is carried by a certain geometric piece of the degeneration which is itself CY, namely the minimal lc center.



Thm 1. Period images have canonical Hodge-theoretic compactifications which are universal wrt the Griffiths bundle.

$\Searrow$

Semiampleness of Griffiths bundle (step in the proof of Thm 1)

$\Nearrow_{GGR}$

Thm 3. Hodge bundles are semiample iff they are integrable and have torsion combinatorial monodromy.

Thm 2. CY moduli spaces have canonical Hodge-theoretic compactifications which are universal wrt the Hodge bundle.

$+$

b-semiampleness

Thm 4. The Hodge bundle of the middle cohomology of a family of klt CY pairs is integrable and has torsion combinatorial monodromy.

Thm 3, Step 1: make the topological space

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(Y, D) proper log smooth, $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on Y°. Then M_Y is semiample if and only if it is **(∗) integrable** *and* **(∗∗) has torsion combinatorial monodromy***.*

Let R be the equivalence relation on Y of being connected by chains of M_Y-degree zero curves.

Lemma

R is a proper algebraic equivalence relation. In particular, $Z = Y/R$ exists as a reasonable topological space.

Moreover, natural stratification of (Y, D) descends to Z—that is, Z has a stratification s.t. inverse images of strata Z_S are unions Y_S of strata of Y.

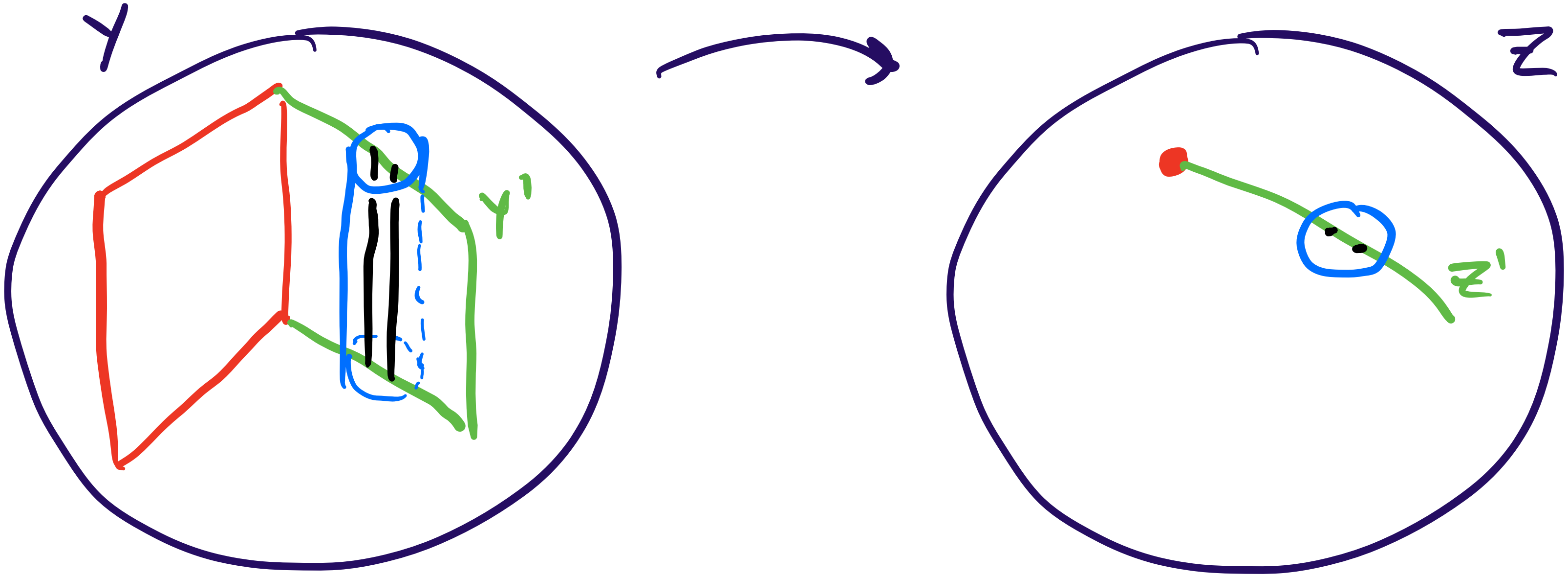
Key: By BBT, **each stratum** Z_S is algebraic and M_Y descends amply to it. Here we use $(*) + (**)$.

Thm 3, Step 2: locally make **some** sections of M_Y

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(Y, D) proper log smooth, $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on Y°. Then M_Y is semiample if and only if it is **$(*)$ integrable** *and* **$(**)$ has torsion combinatorial monodromy**.

Near any point on a stratum Z_S, there are local sections separating fibers **above the stratum** Z_S.

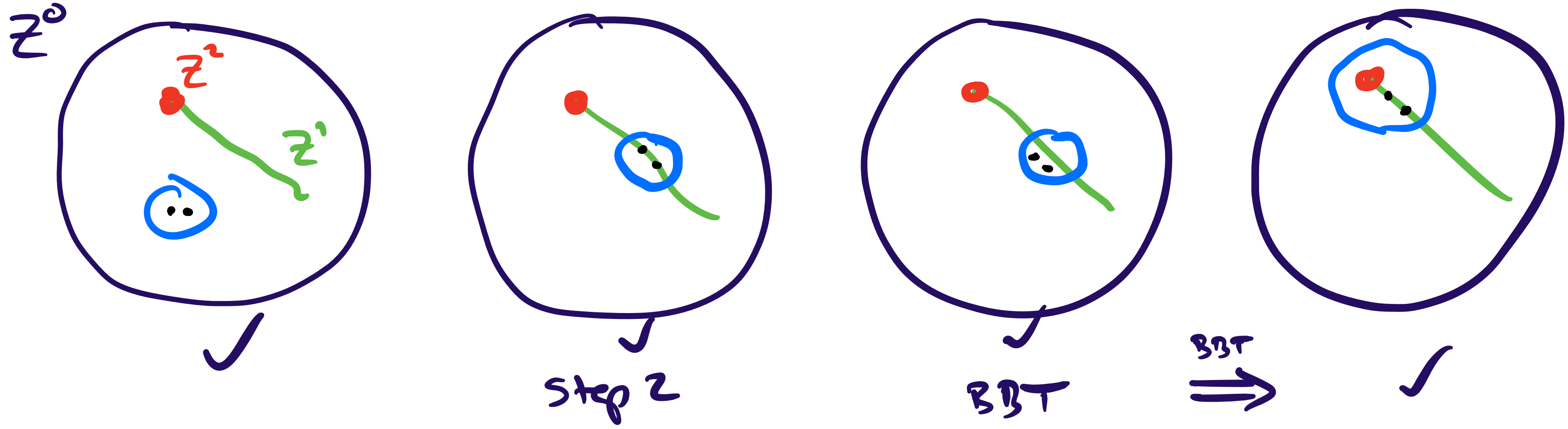


Thm 3, Step 3: inductively glue and algebraize $Z = Y/R$

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(Y, D) proper log smooth, $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on Y°. Then M_Y is semiample if and only if it is **$(*)$ integrable** *and* **$(**)$ has torsion combinatorial monodromy**.

Local sections from Step 2 **DO NOT** give Z an analytic structure!



$Z^{\leqslant i}$ algebraic $\overset{\text{BBT}}{\Rightarrow}$ **global** sections of M_Y separate fibers above $Z^{\leqslant i}$

$\overset{\text{Step 2}}{\Rightarrow}$ $Z^{\leqslant i+1}$ **definable** analytic $\overset{\text{BBT}}{\Rightarrow}$ $Z^{\leqslant i+1}$ algebraic

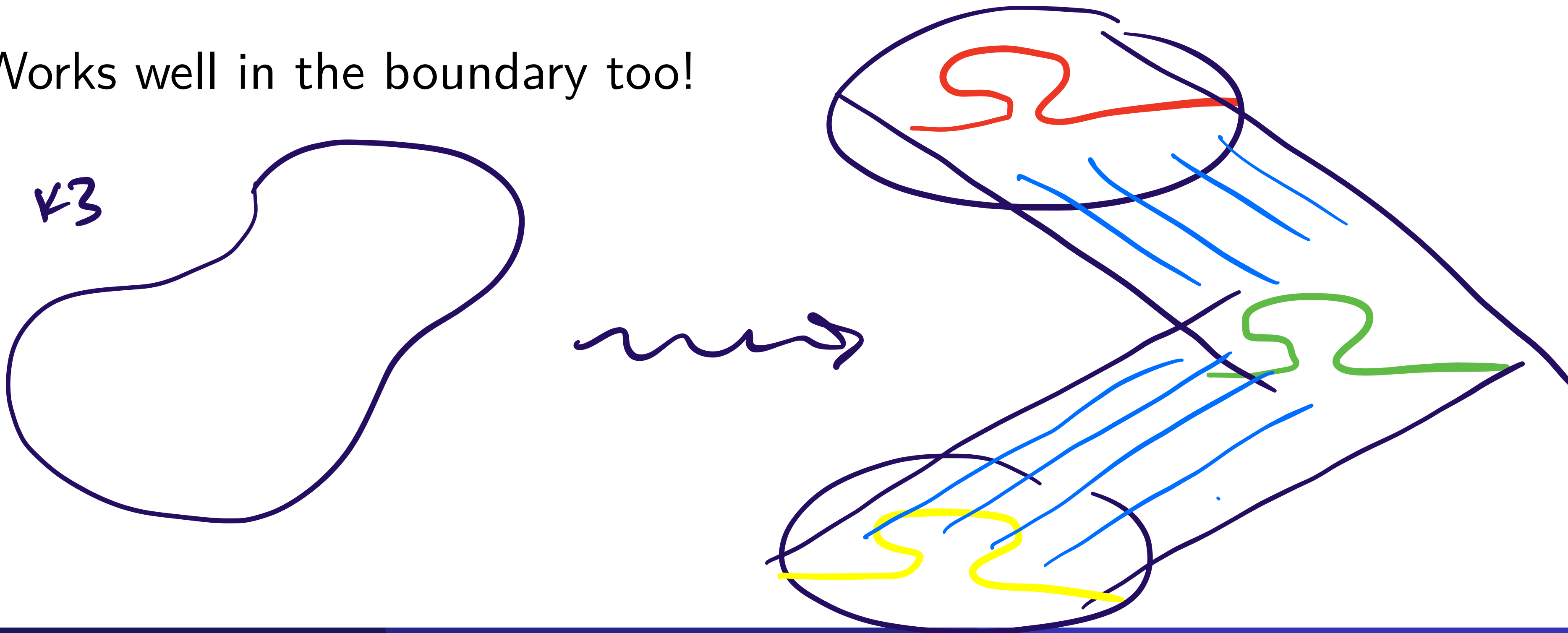
Thm 4: minimal lc centers

Theorem 4 (B–Filipazzi–Mauri–Tsimerman)

(Y, D) proper log smooth, $(V, F^\bullet V)$ the polarizable CY $\mathbb{Z}$-VHS on Y° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

For a lc fibration $\pi : (X, \Delta) \to Y$, a minimal lc center dominating Y (="source") carries the $\mathbb{Q}$-closure V^{tr} of the Hodge bundle.

Key. Works well in the boundary too!



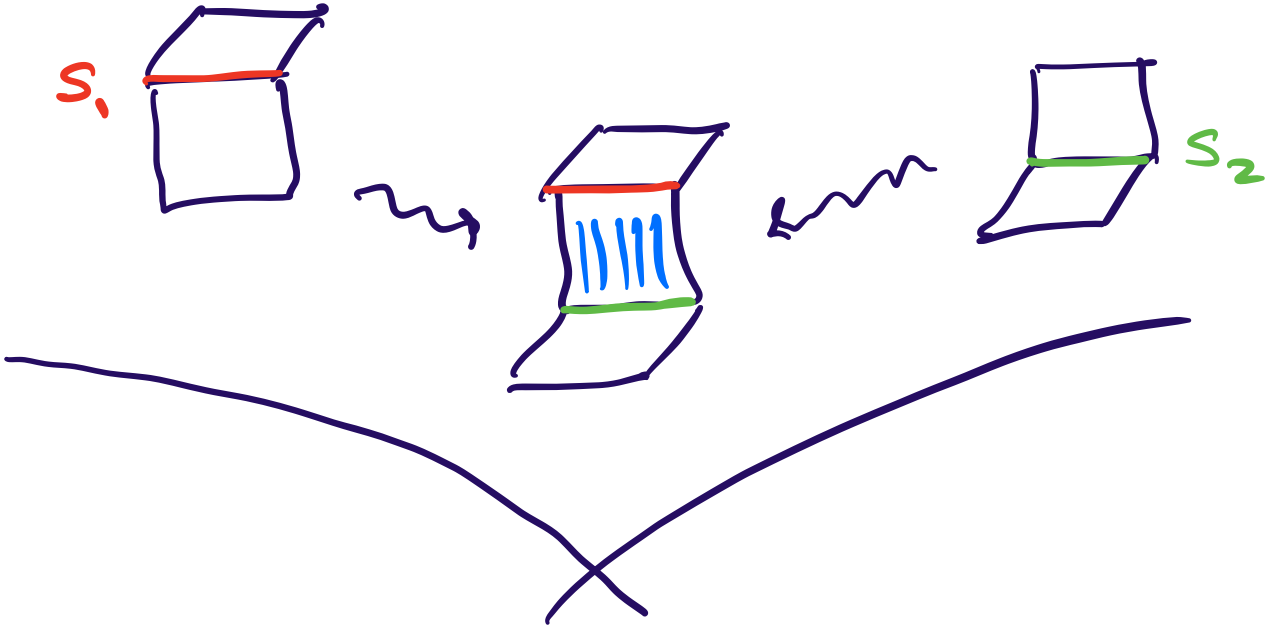
Thm 4: integrability

Essentially a result of Ambro.

Idea. For CY pairs, the period map of the Hodge bundle is immersive on the deformation space.

So if the Hodge bundle is trivial along a transcendental curve, the source must vary trivially, and this is an **algebraic** condition.

Thm 4: torsion combinatorial monodromy



Problem. Source is not unique.

BUT Kollár's $\mathbb{P}^1$-linking $\Rightarrow$ V^{tr}s at node are glued via birational identification of sources $S_1 \simeq S_2$.

$$\operatorname{img}\left(\operatorname{Bir}(S_1, S_2) \to \operatorname{Hom}(H^0(\omega_{S_1}), H^0(\omega_{S_2})\right) < \infty$$

Thanks!