Baily–Borel compactifications of period images and the *b*-semiampleness conjecture

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The classical story: $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$

(1) (Satake '56) $A_g^{\mathrm{SBB}} = A_g \sqcup A_{g-1} \sqcup \cdots \sqcup A_1 \sqcup A_0$.

(2) (Baily '58) $A_g^{\mathrm{SBB}} =$ Proj(graded ring of automorphic forms).

(3) modular interpretation.

$$\begin{array}{ccccc} Z^\circ & \hookrightarrow & Z & \hookleftarrow & Z_0 \\ \downarrow & & \downarrow & & \downarrow \\ C^\circ & \hookrightarrow & C & \ni & 0 \end{array}$$

- Z°/C° a family of abelian varieties
- Z/C is the (semistable) Neron model
- Identity comp. of Z_0 is a semi-abel. variety
- Compact part is limiting point in boundary

(4) natural polarization. $\pi : Z \to \mathcal{A}_g$ the universal family, $L_{\mathcal{A}_g} = \det \pi_* \Omega_{Z/\mathcal{A}_g}$ extends to an ample bundle $L_{A_g^{\mathrm{SBB}}}$ (up to a power).

The classical story: $\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$

(5) <u>universality</u>. (Borel '72)

$$\begin{array}{ccc} X^\circ := X\backslash D & \hookrightarrow & X \\ \downarrow f & & \downarrow \overline{f} \\ \mathcal{A}_g & \longrightarrow & A_g^{\mathrm{SBB}} \end{array}$$

- (X, D) log smooth
- $\overline{f}^* L_{A_g^{\mathrm{SBB}}} \cong L_X$

(5') Even have (5) in the analytic category, i.e.
$(X, D) = (\Delta^k, \text{coordinate hyperplanes})$

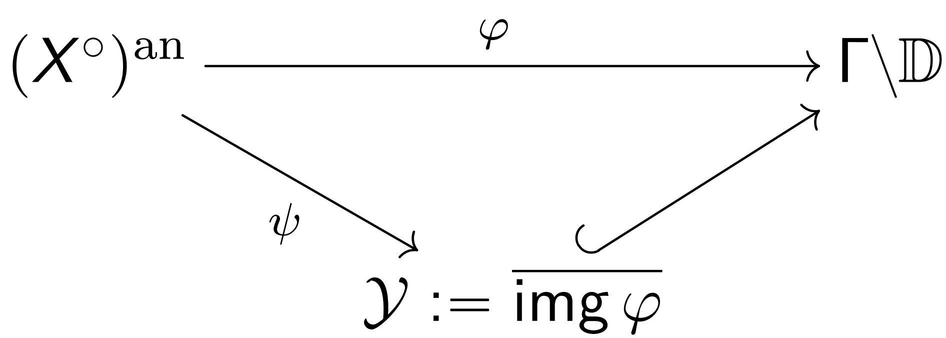
(Satake '60, Baily–Borel '66) Generalized to arbitrary arithmetic locally symmetric varieties.

Period maps

Let (X, D) log smooth proper

$\pi : Z^\circ \to X^\circ$ be a smooth projective family,

$V = R^k \pi_* \mathbb{Z}_{(Z^\circ)^{\mathrm{an}}}$ equipped with its polarizable $\mathbb{Z}$-VHS (filtration $F^\bullet V$ on $\mathcal{O}_{X^{\mathrm{an}}} \otimes_{\mathbb{C}_{X^{\mathrm{an}}}} V$).



Question (Griffiths ’70).

(A) Is $\mathcal{Y}$ algebraic?

(B) Is Griffiths bundle $L_{\mathcal{Y}} := \bigotimes_p \det F^p V$ algebraic? Ample?

(C) Is there a $\mathcal{Y}^{\mathrm{BB}}$?

Main theorem 1

Question (Griffiths '70).

(A) Is $\mathcal{Y}$ algebraic?

(B) Is $L_{\mathcal{Y}} := \bigotimes_p \det F^p V$ algebraic? Ample?

Theorem (B–Brunebarbe–Tsimerman '23)

$\left((X^\circ)^{\mathrm{an}} \xrightarrow{\psi} \mathcal{Y}\right) = \left(X^\circ \xrightarrow{f} Y\right)^{\mathrm{an}}$ and $L_{\mathcal{Y}} = (L_Y)^{\mathrm{an}}$ all algebraic, L_Y ample.

(C) Is there a $\mathcal{Y}^{\mathrm{BB}}$?

Theorem 1 (B–Filipazzi–Mauri–Tsimerman)

- $B_Y := \bigoplus_k H^0_{mg}(Y, L_Y^k)$ *is finitely generated.*
- $Y^{\mathrm{BB}} := \operatorname{Proj} B_Y$ *is projective compactification of* Y *to which* L_Y *extends amply and universally, as in (5) (even (5')).*

Theorem 1 (B–Filipazzi–Mauri–Tsimerman)

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Corollary

Any moduli space with a local Torelli theorem has a canonical minimal Hodge-theoretic compactification.

Remarks

- Y^{BB} is stratified by subvarieties with quasifinite period maps—the ones coming from the associated graded of the limit mixed Hodge structures.
- Lots of previous work of Green–Griffiths–Laza–Robles and Green–Griffiths–Robles, including some special cases. Green–Griffiths–Robles establish key ingredient of our proof.

Other semipositive line bundles

Let $(V, F^\bullet V)$ on X° be a polarizable $\mathbb{Z}$-VHS.

Theorem 1 implies $L_X = \bigotimes_p \det F^p V$ is semiample.

In fact each $\det F^p V$ is semipositive...NOT always semiample...but SOMETIMES!

If the deepest part of $F^\bullet V$ is a line bundle, we call V a CY $\mathbb{Z}$-VHS, and call $M_{X^\circ} = F^{\mathrm{top}} V$ the *Hodge bundle*.

Example. $\det F^p V = \mathrm{Hodge}\left(\bigwedge^{\mathrm{rk}\, F^p V} V\right)$.

Example. $\mathrm{Griffiths}\,(V) = \mathrm{Hodge}\left(\bigotimes_p \bigwedge^{\mathrm{rk}\, F^p V} V\right)$.

Example. If $\pi : Z^\circ \to X^\circ$ a smooth projective family of Calabi–Yau m-folds, $V = R^m \pi_* \mathbb{Z}_{(Z^\circ)^{\mathrm{an}}}$. Then $M_{X^\circ} = \pi_* \omega_{Z^\circ / X^\circ}$.

When is the Hodge bundle M_X semiample?

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

Let (X, D) be a proper log smooth algebraic space and $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on X°. Then M_X is semiample if and only if it is **integrable** *and* **has torsion combinatorial monodromy***.*

Integrability. If M_X is flat on some analytic germ T, it is flat on the Zariski closure T^{Zar}.

Automatic for the Griffiths bundle

Torsion combinatorial monodromy. If M_X is numerically trivial on a connected curve, it is torsion.

Theorem (Green–Griffiths–Robles)

The Griffiths bundle has torsion combinatorial monodromy.

Families of Calabi–Yau varieties

Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ the polarizable CY $\mathbb{Z}$-VHS on X° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

Corollary

Coarse spaces of moduli stacks $\mathcal{Y}$ of smooth polarized Calabi–Yau varieties have canonical Baily–Borel compactifications Y^{BBH}:

- *$C_Y := \bigoplus_k H^0_{mg}(Y, M_Y^k)$ is finitely generated.*
- *$Y^{\mathrm{BBH}} := \operatorname{Proj} C_Y$ is projective compactification of Y to which* **the Hodge bundle** *M_Y extends amply and universally, as in (5/5').*

- Also works for klt Calabi–Yau pairs.
- Implies the *b*-semiampleness conjecture of Mori, Kawamata, Shokurov, Ambro, and Prokhorov–Shokurov. Partial past results of Ambro, Lazić, Floris,...

Theorem 1 (BFMT)

The image Y of a period map admits a canonical minimal Hodge-theoretic compactification Y^{BB}.

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Semiampleness of Griffiths bundle

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Theorem 2 (BFMT)

The Hodge bundle M_X of a polarizable CY $\mathbb{Z}$-VHS is semiample if and only if it is integrable and has torsion combinatorial monodromy.

Canonical Hodge-theoretic compactifications Y^{BBH} of CY moduli which are minimal wrt the Hodge bundle

Theorem 3 (BFMT)

The Hodge bundle of the middle cohomology of a family of klt CY pairs is integrable and has torsion combinatorial monodromy.

Thm 2, Step 1: make the topological space

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on X°. Then M_X is semiample if and only if it is **(∗) integrable** *and* **(∗∗) has torsion combinatorial monodromy**.

Let R be the equivalence relation on X of being connected by chains of M_X-degree zero curves.

Lemma

R is a proper algebraic equivalence relation. In particular, $Y = X/R$ exists as a reasonable topological space.

Moreover, natural stratification of (X, D) descends to Y—that is, Y has a stratification s.t. inverse images of strata Y_S are unions X_S of strata of X.

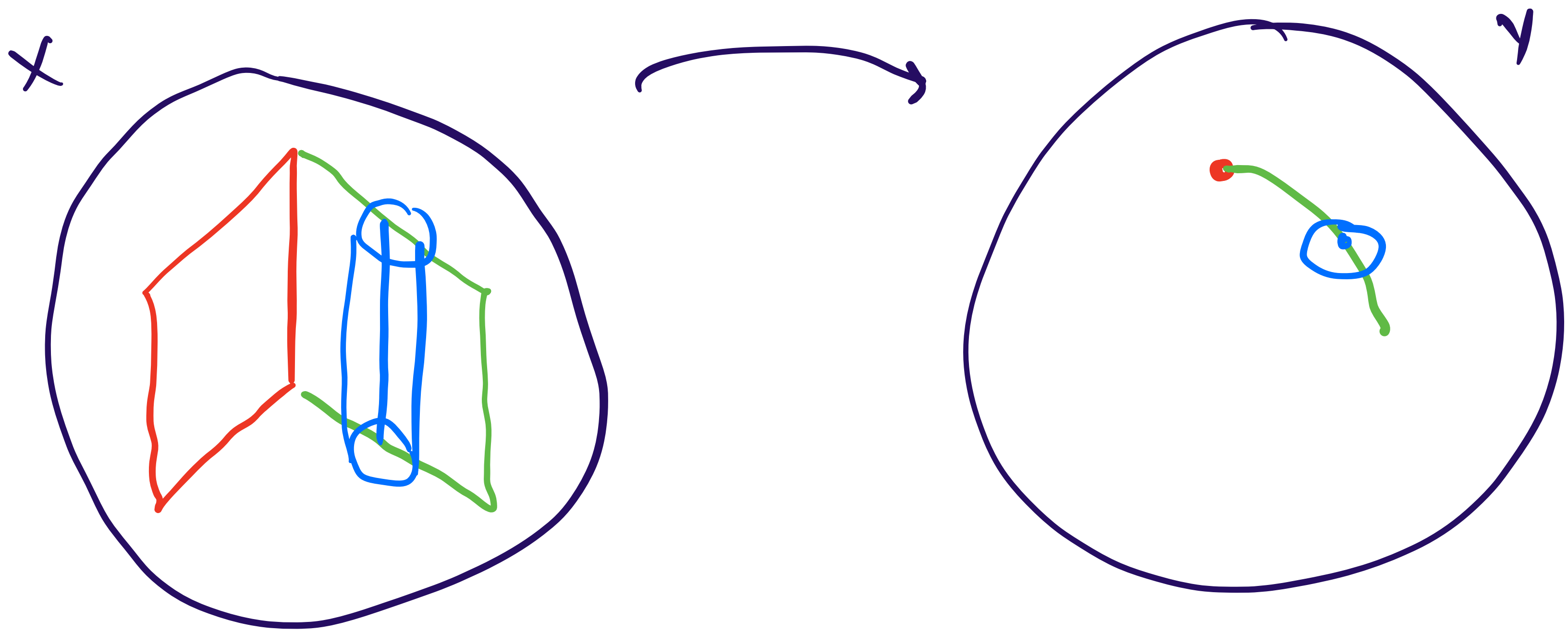
Key: By BBT, **each stratum** Y_S is algebraic and M_X descends amply. Here we use $(*) + (**)$.

Thm 2, Step 2: locally make **some** sections of M_X

Theorem 2 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ a polarizable CY $\mathbb{Z}$-VHS on X°. Assume the Hodge bundle M_X is $(*)$ **integrable** *and $(**)$* **has torsion combinatorial monodromy**. *Then M_X is semiample.*

Near any point on a stratum Y_S, there are local sections separating fibers **over the stratum** Y_S.



Thm 2, Step 3: inductively glue and algebraize $Y = X/R$

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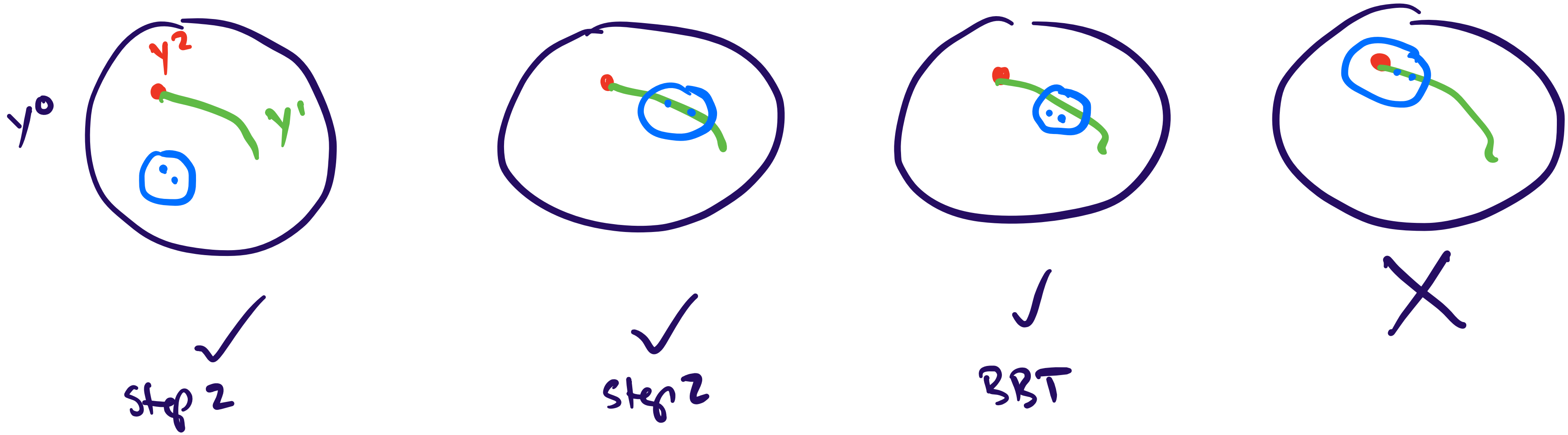
Local sections from Step 2 **DO NOT** give Y an analytic structure!



$Y^{\leqslant i}$ algebraic $\overset{\text{BBT}}{\Rightarrow}$ **global** sections of M_Y separate points

$\Rightarrow Y^{\leqslant i+1}$ **definable** analytic $\overset{\text{BBT}}{\Rightarrow}$ $Y^{\leqslant i+1}$ algebraic

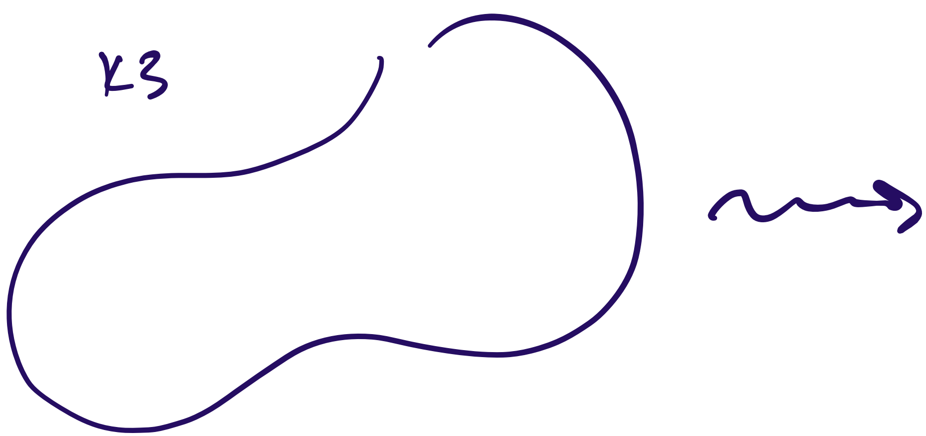
Thm 3: minimal lc centers

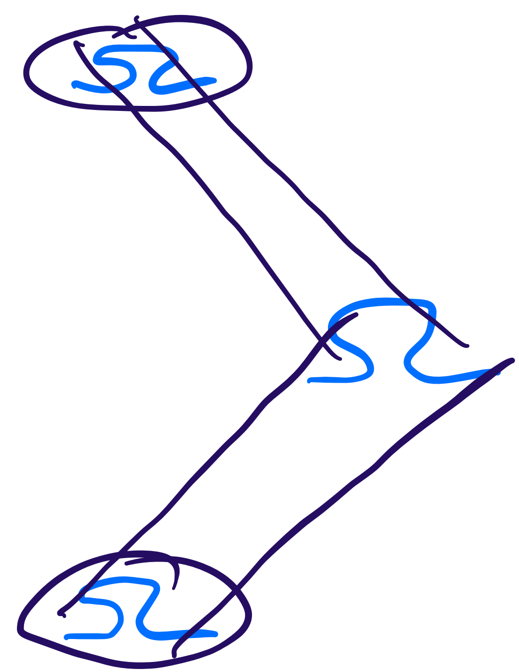
Theorem 3 (B–Filipazzi–Mauri–Tsimerman)

(X, D) proper log smooth, $(V, F^\bullet V)$ the polarizable CY $\mathbb{Z}$-VHS on X° coming from the middle cohomology of a family of klt CY pairs. Then the Hodge bundle is integrable and has torsion combinatorial monodromy.

For a lc fibration $\pi : (Z, \Delta) \to X$, a minimal lc center dominating X (="source") carries the $\mathbb{Q}$-closure V^{tr} of the Hodge bundle.

Key. Works well in the boundary too!





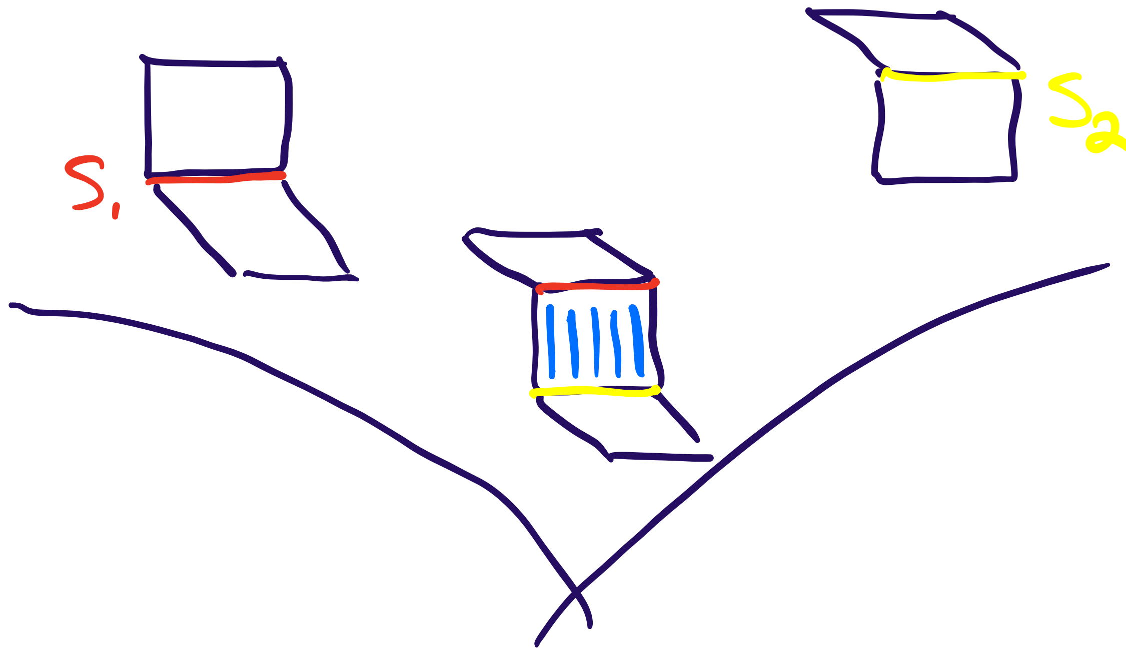
Thm 3: integrability

Essentially a result of Ambro.

Idea. For CY pairs, the period map of the Hodge bundle is immersive on the deformation space.

So if the Hodge bundle is trivial along a transcendental curve, the source must vary trivially, and this is an **algebraic** condition.

Thm 3: torsion combinatorial monodromy



Problem. Source is not unique.

BUT Kollár's $\mathbb{P}^1$-linking $\Rightarrow$ V^{tr}s at node are glued via birational identification of sources $S_1 \simeq S_2$.

$$\mathrm{img}\left(\mathrm{Bir}(S_1, S_2) \to \mathrm{Hom}(H^0(\omega_{S_1}), H^0(\omega_{S_2})\right) < \infty$$

Thanks!